2.4 Bose-Einstein Condensation

Einstein 1924
Bose 1925
London 1938

Basic idea of Fritz London:

dissipation-less motion $\longleftrightarrow$ macroscopic wave function

a) Ideal Bose gas

non-interacting Bose gas (rough approximation for liquid He)

let's consider: 1 cm^3 cube of liquid ^{4}He $\triangleq 10^{22}$ atoms with mass *m*

eigenstates for free particles in a cube:

$$E_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2 \qquad \text{with} \qquad n^2 = n_x^2 + n_y^2 + n_z^2$$

$T = 0 \longrightarrow$ all atoms are in the ground state E_{111} trivial !

But at finite temperatures?

2.4 Bose-Einstein Condensation

consider energy difference between ground state and first excited state

$$\Delta E/k_\mathrm{B} = (E_{211} - E_{111})/k_\mathrm{B} \approx 2 \times 10^{-14}\,\mathrm{K}$$

if Boltzmann statistics would hold ➡ no condensate at 1 K!!!

however, Bose-Einstein distribution is relevant here

$$f(E,T) = \frac{1}{\mathrm{e}^{(E-\mu)/k_\mathrm{B}T} - 1}\,.$$

chemical potential $\mu = \dfrac{\partial F}{\partial N}$

what we know: $\mu < E_{111}$ $\longrightarrow$ otherwise, negative occupation

$\mu \neq 0$ $\longrightarrow$ since particle number conserved

2.4 Bose-Einstein Condensation

Occupation of ground state $E_{111} = 0$

$$f(0,T) = \frac{1}{\mathrm{e}^{-\mu/k_\mathrm{B}T} - 1}$$ ➡ occupation depends critically on μ

$f(0, T \to 0) \to \infty$ if $\mu \to 0$ faster than $T \to 0$ $\left(\frac{1}{e^0 - 1} \to \infty \right)$

What is the temperature dependence of $\mu(T)$?

for this let us consider a real, but non-interacting gas

$$\mu = -k_\mathrm{B} T \ln \left(\frac{V_\mathrm{A}}{V_\mathrm{Q}} \right)$$

quantum volume $V_\mathrm{Q} = \left(\frac{h}{\sqrt{2\pi m k_\mathrm{B} T}} \right)^3 = \lambda_\mathrm{B}^3$

thermal de Broglie wavelength

For ^{4}He $\longrightarrow$ $\lambda_\mathrm{B}^3 =$ (8.7 Å)^3 at 1 K

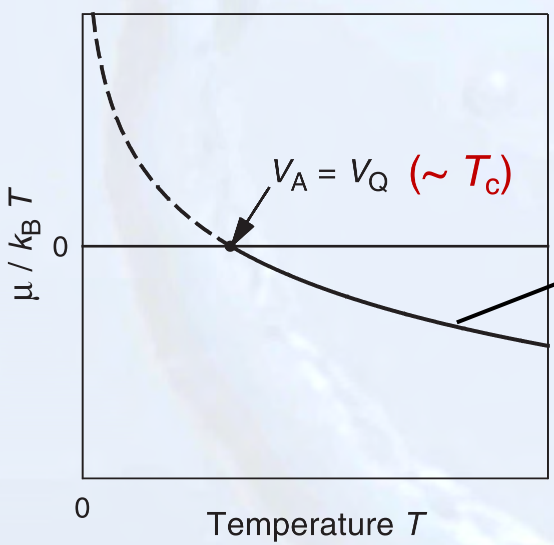
$V_\mathrm{A} = V/N =$ (3.8 Å)^3 in comparison

2.4 Bose-Einstein Condensation

for $T \to 0$ ➡ $V_\mathrm{Q} \to \infty$

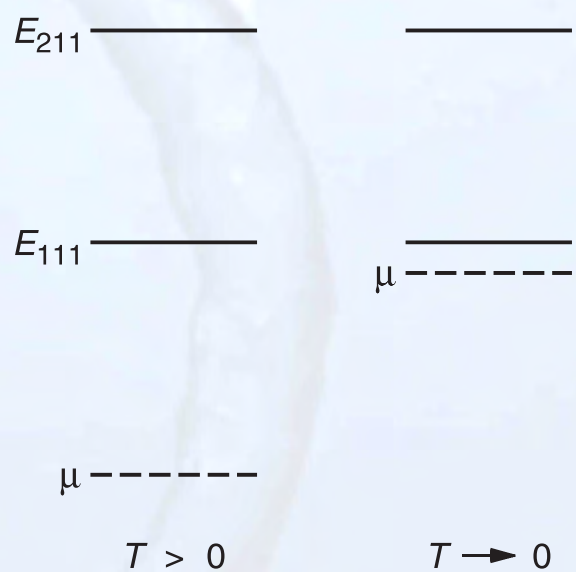
at sufficiently low temperatures $V_\mathrm{A} = V_\mathrm{Q}$ **!**

$$\ln\left(\frac{V_\mathrm{A}}{V_\mathrm{Q}}\right) \to 0 \quad ➡ \quad \mu \to 0$$



classical regime
μ is negative

He gas at 300 K:
$\mu/k_\mathrm{B} = -3800\,\mathrm{K}$



➡ this means $|\mu|$ becomes smaller than $\Delta E/k_\mathrm{B} = (E_{211} - E_{111})$ at finite T **!**

2.4 Bose-Einstein Condensation

Calculation of μ: how large is μ at 1K? (revers argument)

for $T \to 0$ ➡ $f_{111} \to N$

$$\lim_{T\to 0} f(0,T) = N_0(T) = \lim_{T\to 0}\left(\frac{1}{e^{-\mu/k_\mathrm{B}T} - 1}\right)$$

$E_{111} = 0$, ground state

$$\approx \lim_{T\to 0}\left(\frac{1}{1-\mu/(k_\mathrm{B}T)+\ldots-1}\right) \approx -\frac{k_\mathrm{B}T}{\mu}$$

$$\boxed{\mu = -\frac{k_\mathrm{B}T}{N_0}}$$ close to $T = 0$

➡ at $T = 1\ \mathrm{K} \longrightarrow \mu/k_\mathrm{B} \approx 10^{-22}\ \mathrm{K}$!

2.4 Bose-Einstein Condensation

Calculation of N_0 and N_e :

number of particles in excited states

$$\sum_i f(E_i, T) = N = N_0(T) + N_\mathrm{e}(T)$$

$$= N_0(T) + \int_0^\infty D(E)\, f(E,T)\,\mathrm{d}E$$

density of states for free particles without $D(0)$

density of states for free particles $E_k \propto k^2$

$$D(E) = \frac{V(2m)^{3/2}\sqrt{E}}{4\pi^2\hbar^3}$$

with $E/k_\mathrm{B}T = x$ and $|\mu| \ll \Delta E$ $\longrightarrow$ $\exp[(E-\mu)/k_\mathrm{B}T] \approx \exp(E/k_\mathrm{B}T)$

$$\Rightarrow \quad N = N_0 + \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(k_\mathrm{B}T)^{3/2} \underbrace{\int_0^\infty \frac{\sqrt{x}}{\mathrm{e}^x - 1}\,\mathrm{d}x}_{\Gamma(5/2)\times\zeta(5/2)\approx 1.783}$$

2.4 Bose-Einstein Condensation

with $V_\mathrm{Q} = \left(\dfrac{h}{\sqrt{2\pi m k_\mathrm{B} T}}\right)^3 = \lambda_\mathrm{B}^3$

$$N \approx N_0 + 2.6\,\frac{V}{V_\mathrm{Q}}$$

$$\boxed{N_0 = N - 2.6\,\frac{V}{V_\mathrm{Q}}}$$

Interpretation

as long as $2.6\,\underbrace{\frac{V}{V_\mathrm{Q}}}_{NV_\mathrm{A}} \ll 10^{22}$, which means that the de Broglie wavelength is

significantly larger as an atom ➡ condensation !

factor $\sqrt[3]{2.6} = 1.37$

2.4 Bose-Einstein Condensation

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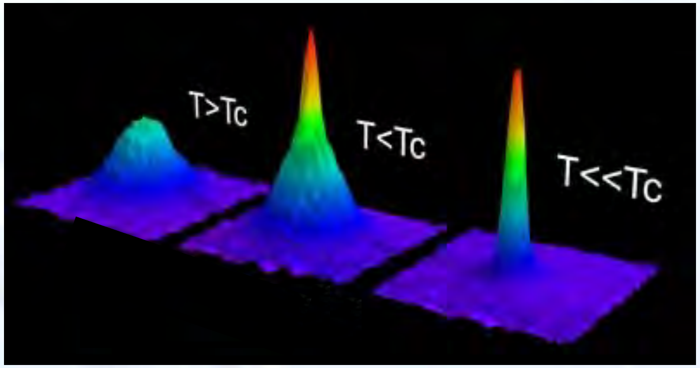
- $T = 0 \longrightarrow N_0 = N$ trivial !
- $0 < T < T_\mathrm{c} \longrightarrow N_0$ still macroscopically large!
- $N_\mathrm{e} \triangleq$ normalfluid component

comment:
λ_B^3 must not be as large as the vessel as proposed by London

2.4 Bose-Einstein Condensation

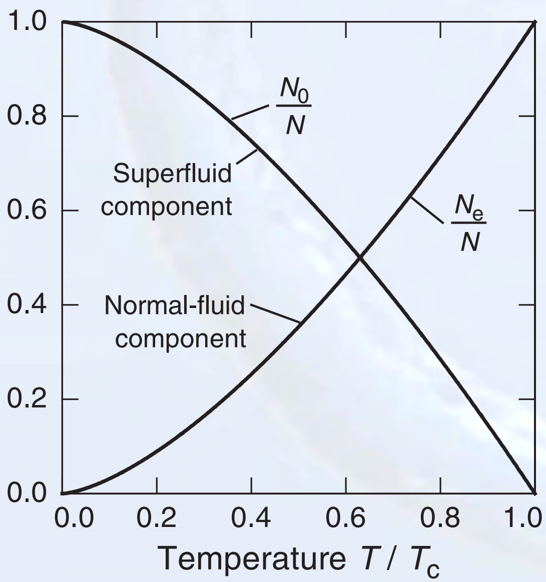
What is the value of the condensation temperature?

$$\left.\begin{aligned} N_0(T_c) &= 0 \\ N_e(T_c) &= N \end{aligned}\right\} \qquad T_c = \frac{2\pi\hbar^2}{k_B m}\left(\frac{N}{2.6V}\right)^{2/3}$$



Bose Einstein condensate of atomic gas

He
- gas $T_c \approx 0.5\,\mathrm{K}$, but boiling point is at $4.2\,\mathrm{K}$
- liquid $T_c = 3.1\,\mathrm{K}$, works well in comparison to $T_\lambda = 2.17\,\mathrm{K}$



$$\frac{N_e}{N} = \left(\frac{T}{T_c}\right)^{3/2}$$

the condensation of a normal gas in real space corresponds to the Bose-Einstein condensation in momentum space, which means all atoms have the same wave vector and are strongly correlated.

2.4 Bose-Einstein Condensation

b) Interacting Bose "gas" (He)

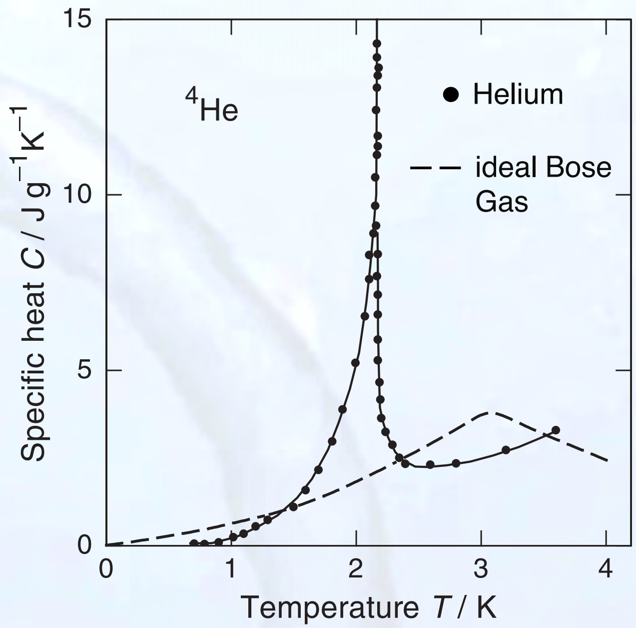
specific heat

C_V → ideal Bose gas
C_V → experimental results

➡ interactions are important

→ depletion of the ground state

→ collective excitation
(has first been proposed by Bogoliubov 1947)

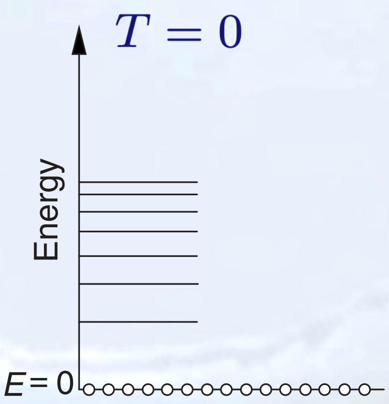


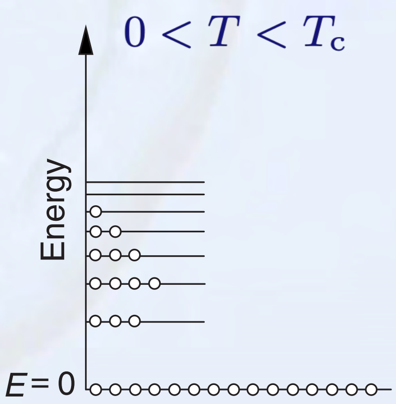
ideal Bose gas

$T=0\,,\;\; N_0=N$

$T\neq 0\,,\;\; N_0\leq N$

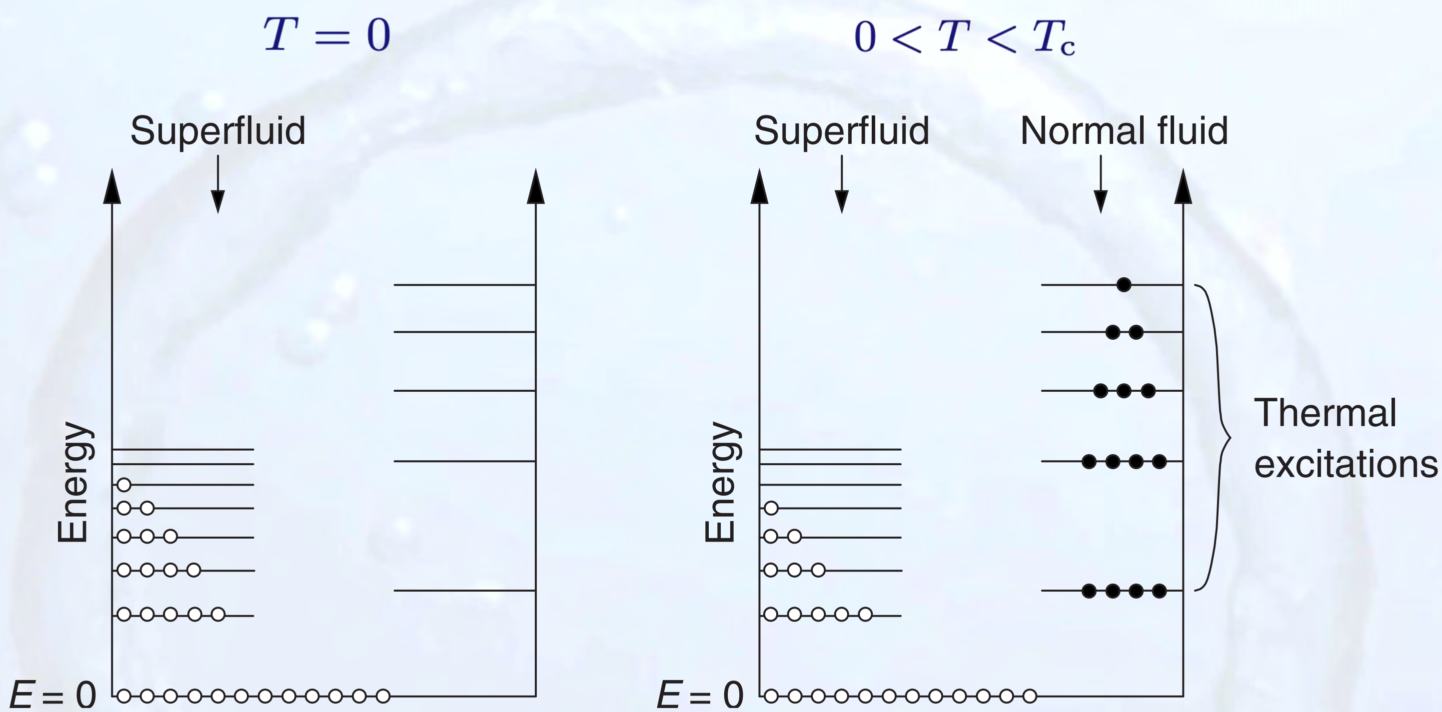
excited atoms





2.4 Bose-Einstein Condensation

interacting Bose gas



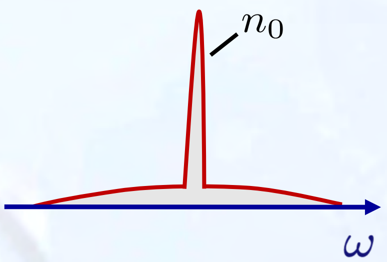
$T = 0\,,\; N_0 < N$: significant number of atoms are not in the ground state

$T \neq 0\,,\; N_0 < N$: in addition, collective excitations, nature of excitations changes

2.4 Bose-Einstein Condensation

Experimental determination of the condensate

there is no direct way to measure the condensate fraction: $N_0/N = n_0$



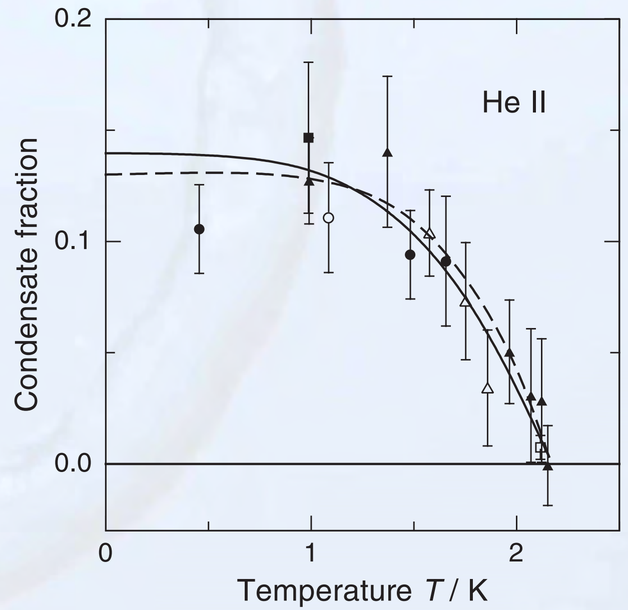
a) neutron scattering: measuring the dynamic structure factor $S(Q,\omega) \mathrel{\widehat{=}} n(p)$
via inelastic neutron scattering

momentum distribution

b) X-ray scattering: pair correlation function $g(r)$ at transition to superfluid state
becomes broader because of the condensation in momentum space

$$g(r) - 1 = (1 - n_0)^2 \, [g^*(r) - 1]$$

$g(r)$ above



c) surface tension: complicated but possible

➡ condensate fraction for $T \to 0$ just 13 %

ϱ_s is not equal with condensate fraction