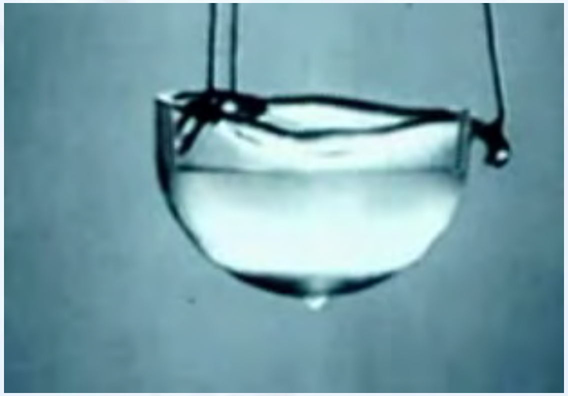
2.3 Properties of He-II described using the two-fluid model

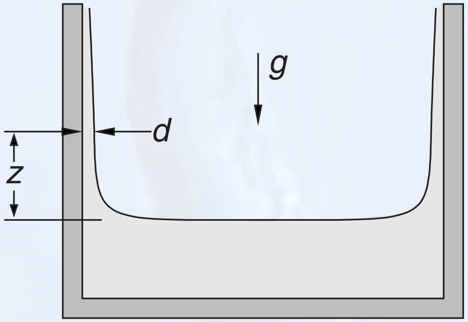
b) Beaker experiments

films are formed with a thickness of ~ 200 Å in saturated vapor pressure also against gravity

let us understand how
comment: the film formation is a “classical” phenomenon



(i) Film formation in saturated vapor



In thermal equilibrium

$$\mu_f = \mu_g = \mu_\ell$$

chemical potential for film (gas and liquid)

gravitational force is compensated by v. Waals forces

$$\Rightarrow \quad \mu_f = \mu_\ell + \underbrace{\mu_{grav} + \mu_{vdW}}_{=0} = \mu_\ell$$

2.3 Properties of He-II described using the two-fluid model

film thinkness:

$$\left.\begin{aligned}\mu_{\rm grav} &= gz \\ \mu_{\rm vdW} &= -\alpha/d^3\end{aligned}\right\} \quad gz - \alpha/d^3 = 0 \quad \curvearrowright \quad \boxed{d = \sqrt[3]{\frac{\alpha}{gz}}}$$

depends on film thickness:

$$\mu_{\rm vdW} = -\alpha/d^3 \quad \text{for} \quad d < 30\,\mathrm{nm}$$

$$\mu_{\rm vdW} = -\alpha/d^4 \quad \text{for} \quad d > 80\,\mathrm{nm}$$

retardation of potential

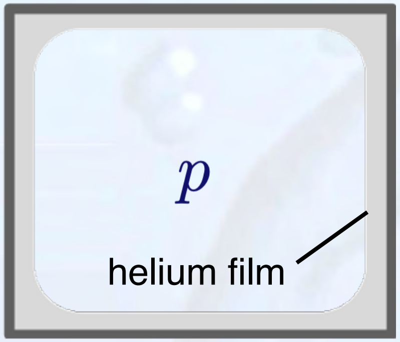
α: atomic polarisability of helium + wall (Hamaker constant)

typical value: $d \sim 20$ nm at $z = 10$ cm

comment: property of superfluidity is unimportant for the film formation and thickness, but for the film flow

2.3 Properties of He-II described using the two-fluid model

(ii) film formation in unsaturated vapor



How does d depend on p ?

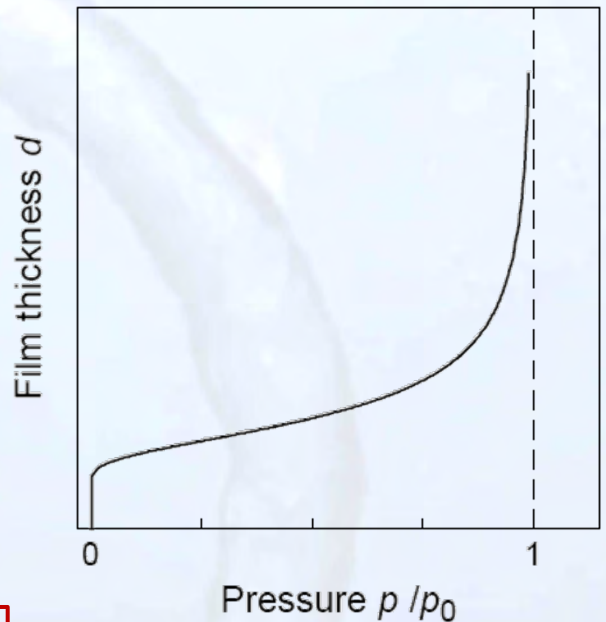
➡ barometric formula

$$\frac{p}{p_0} = \mathrm{e}^{-mgh/k_\mathrm{B}T}$$

$$mgh = -\,k_\mathrm{B}T \ln\left(\frac{p}{p_0}\right)$$

➡ $$\frac{\alpha}{d^3} = \frac{k_\mathrm{B}T}{m_4} \ln\left(\frac{p_0}{p}\right)$$

➡ $$d = \sqrt[3]{\frac{\alpha\, m_4}{k_\mathrm{B}T(\ln p_0 - \ln p)}}$$



- decrease of pressure ➡ decrease of film thickness
- in practice: thicknesses of sub-mono layers are possible and realized

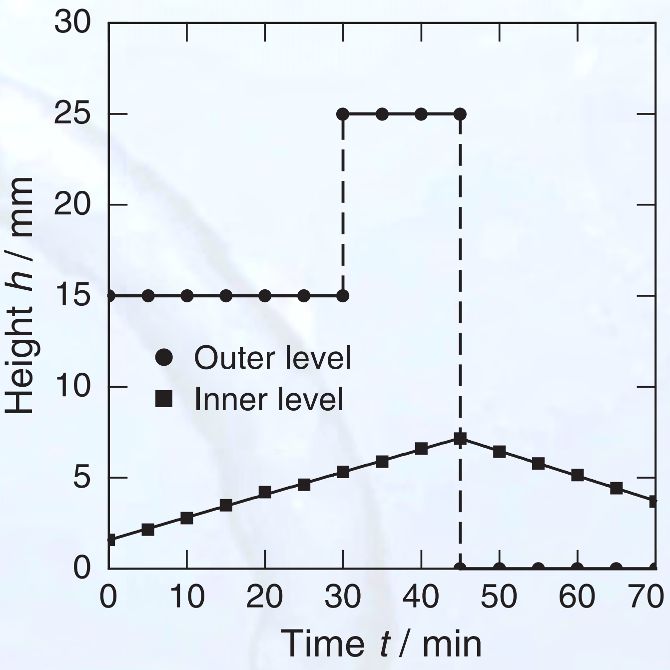
⬇

investigation of superfluidity with third sound: onset of superfluidity at $n >$ 2.1 layers

2.3 Properties of He-II described using the two-fluid model

now back to the film flow:

- films are formed
- ϱ_s is moving without friction
- equalizing the chemical potential is driving force
- const. rate $\triangleq$ critical velocity

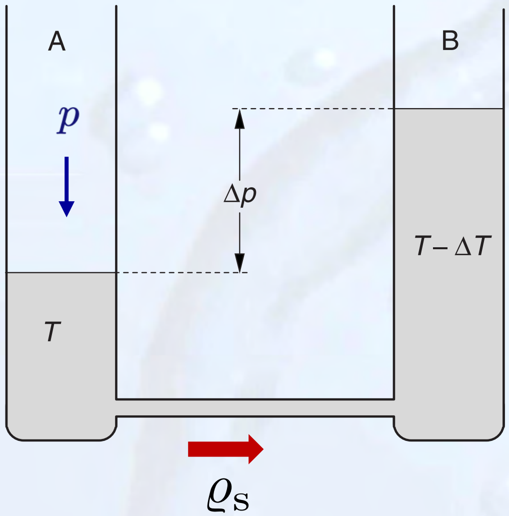


Interesting question: ϱ_s flows with $S = 0$! ➡ rest should warm up and helium flowing into a vessel should have $T = 0$!

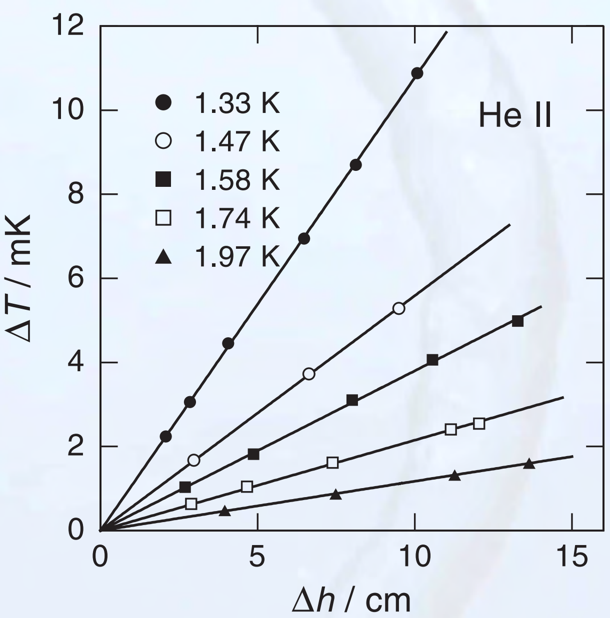
but thermal equilibrium via gas phase

2.3 Properties of He-II described using the two-fluid model

c) Thermomechanical effect



$+\,\varrho_\mathrm{s}$ → cooling in B
$-\,\varrho_\mathrm{s}$ → warming in A
} $T_\mathrm{B} < T_\mathrm{A}$



Using (6) in stationary state

$$\frac{\partial \boldsymbol{v}_\mathrm{s}}{\partial t} = S\,\mathrm{grad}\,T - \frac{1}{\varrho}\,\mathrm{grad}\,p = 0$$

↑ in equilibrium nothing flows

$$\frac{\Delta p}{\Delta T} = \varrho S$$

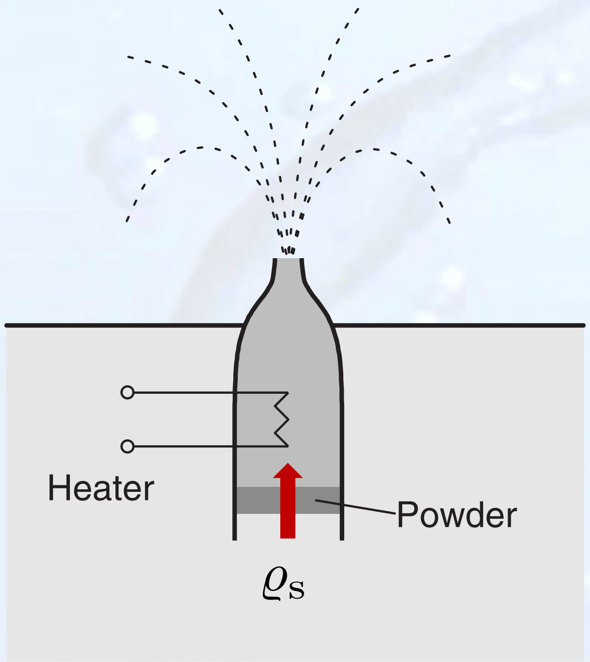
London equation
(H. London 1939)

→ Linear relation between Δp and ΔT

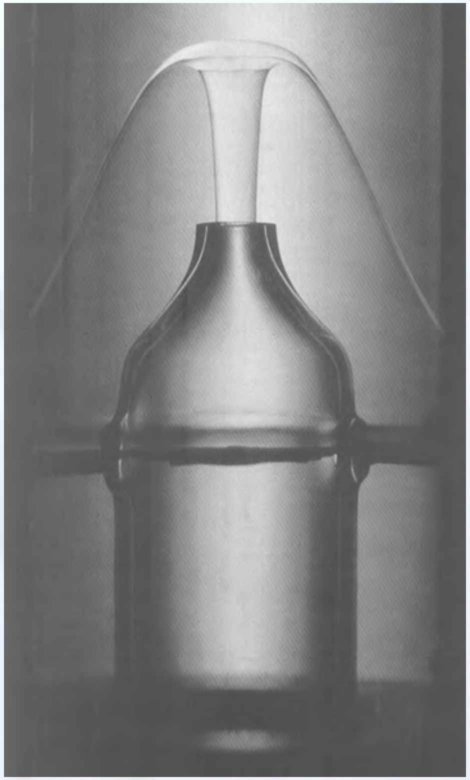
$\Delta h = 2\,\mathrm{cm}$
$T = 1.5\,\mathrm{K}$
} $\Delta T = 1\,\mathrm{mK}$ not very effective cooling

2.3 Properties of He-II described using the two-fluid model

Reverse thermomechanical effect: Fountain effect



$\Delta T \curvearrowright \Delta p$



- heating of helium inside vessel ⟶ ratio of ϱ_n / ϱ_s increases inside the vessel
- the temperature inside is higher than outside
- to equalize the system ϱ_s flows through superleak (compressed powder)
- pressure rises and fountain starts to flow (and flows as long as heater is on)

2.3 Properties of He-II described using the two-fluid model

d) Heat Transport

- in not too small capillaries $v_{\mathrm{n}} \neq 0$
- even in equilibrium ($\Delta p = \varrho S \Delta T$) there is a constant flow of ϱ_{n} from the warm end to the cold end and ϱ_{s} in the opposite direction by "convection"

$\varrho_{\mathrm{n}} \longrightarrow$ cold end
$\varrho_{\mathrm{s}} \longrightarrow$ warm end

} entropy transport $\triangleq$ heat transport

heat transport maximum at 1.8 K where $\varrho_{\mathrm{n}} \approx \varrho_{\mathrm{s}}$

- limited only by the mobility of ϱ_{n} and therefore η_{n}
- viscos mass flow of ϱ_{n} :

$$\dot{V}_{\mathrm{n}} = \frac{\beta}{\eta_{\mathrm{n}}} \frac{\Delta p}{L} \qquad (*)$$

volume rate ($\dot{V}_{\mathrm{n}}$)

$\beta \propto r^4$ for capillaries

$\beta \propto d^3$ for slits

- entropy flow $\dot{V}_{\mathrm{n}} \varrho S \longrightarrow$ heat flow $\dot{Q} = T \dot{V}_{\mathrm{n}} \varrho S \qquad (**)$

$\delta Q = T \delta S$

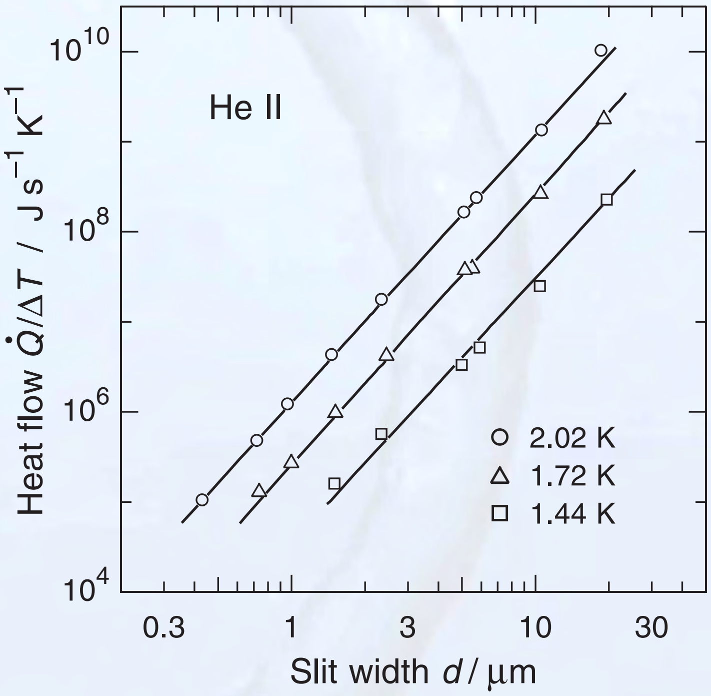
2.3 Properties of He-II described using the two-fluid model

$(*)$ insert in $(**)$ and London equation $(\Delta p = \varrho S \Delta T)$

$$\dot{Q} = \frac{\beta T (\varrho S)^2}{\eta_\mathrm{n} L} \Delta T$$

linear regime

heat flow $\log \dot{Q}/\Delta T$ vs $\log d$



experimental results:

$\dot{Q} \propto \beta \propto d^3$ (as expected)

$\dot{Q}$ rises with T (as expected)

2.3 Properties of He-II described using the two-fluid model

Momentum of heat flow

Heat flow in He-II ⟶ momentum flow $\varrho\,\boldsymbol{v}\cdot\boldsymbol{v}$

momentum flow / volume

resulting pressure acting on a heat source

$$p = \varrho_\mathrm{n} v_\mathrm{n}^2 + \varrho_\mathrm{s} v_\mathrm{s}^2 \qquad (*)$$

no net mass transport (closed vessel)

$$\varrho_\mathrm{n} v_\mathrm{n} + \varrho_\mathrm{s} v_\mathrm{s} = 0 \longrightarrow v_\mathrm{s} = -\frac{\varrho_\mathrm{n}}{\varrho_\mathrm{s}} v_\mathrm{n}$$

insert in $(*)$ ⟶ $p = \dfrac{\varrho_\mathrm{n}\varrho}{\varrho_\mathrm{s}} v_\mathrm{n}^2$

with heat flow / per area $\dfrac{\dot{Q}}{A} = \varrho S T v_\mathrm{n}$ ⟶ $v_\mathrm{n} = \dfrac{\dot{Q}}{A \varrho S T}$

$\dfrac{\dot{V}_\mathrm{n}}{A}$

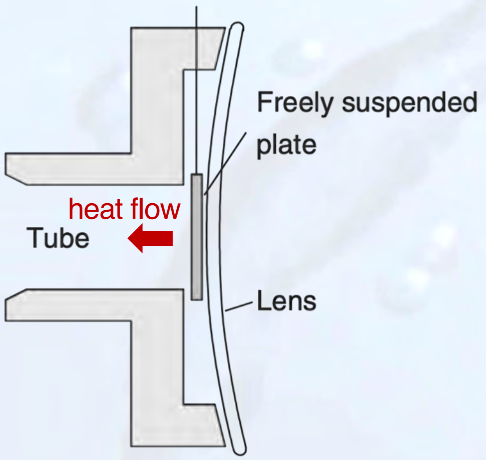
$$p = \frac{\varrho_\mathrm{n}}{\varrho_\mathrm{s}\varrho}\left(\frac{\dot{Q}}{AST}\right)^2$$

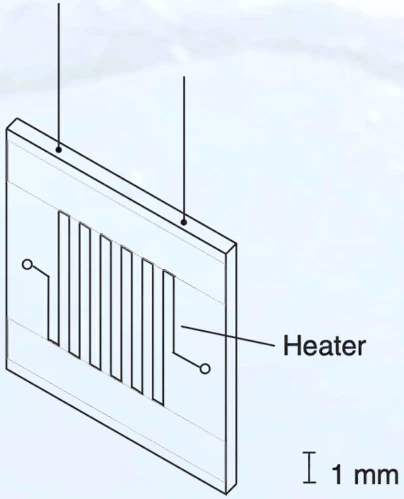
pressure associated with uni-directional heat flow

$$p = \varrho_n v_n^2 + \varrho_s v_s^2 \quad \left(v_s = -\frac{\varrho_n}{\varrho_s} v_n\right)$$
$$= \varrho_n v_n^2 + \varrho_s \left(-\frac{\varrho_n}{\varrho_s} v_n\right)^2$$
$$= \varrho_n v_n^2 \left(1 + \frac{\varrho_n}{\varrho_s}\right)$$
$$= \varrho_n v_n^2 \left(\frac{\varrho_s + \varrho_n}{\varrho_s}\right)$$
$$= \frac{\varrho_n \varrho}{\varrho_s} v_n^2$$

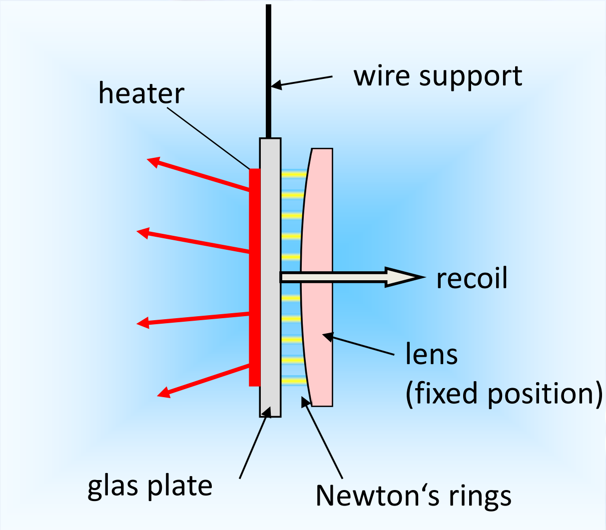
2.3 Properties of He-II described using the two-fluid model

Momentum of heat flow: Measurement

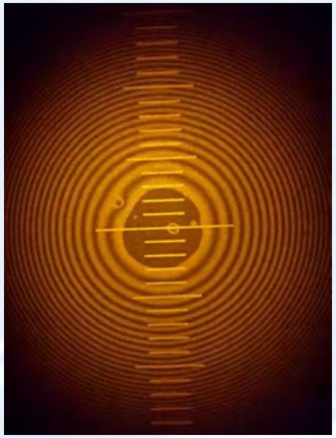




change of distance between glass plate and lens measured by Newton rings ⟶ force



Newton rings



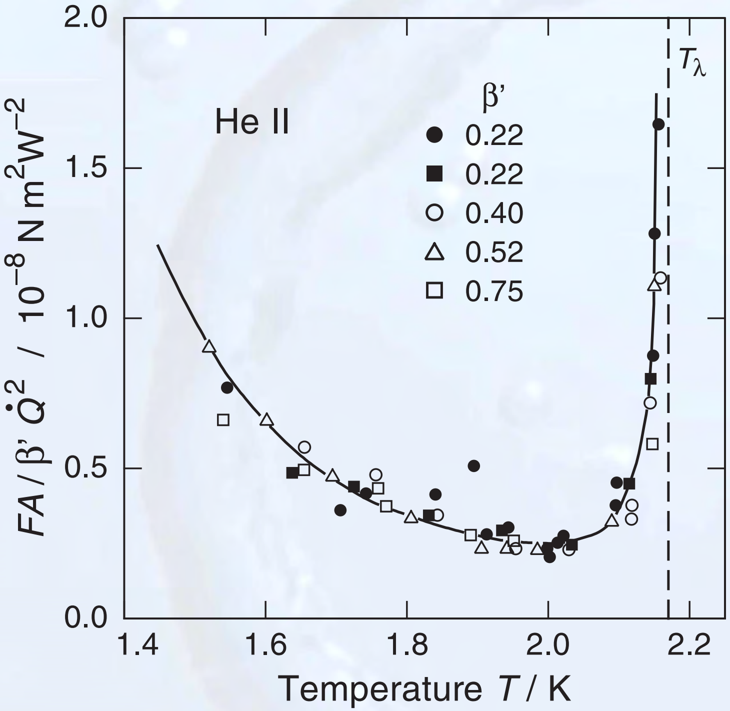
Expected force

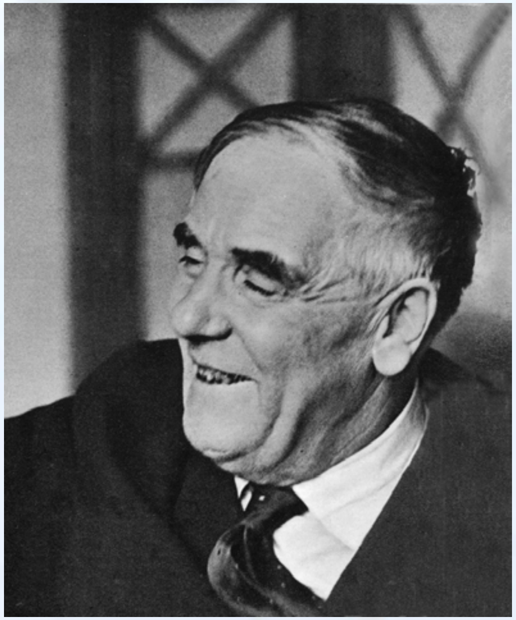
$$F = pA = \beta' \frac{\varrho_\mathrm{n}}{\varrho_\mathrm{s}\varrho A}\left(\frac{\dot{Q}}{ST}\right)^2$$

geometry dependent factor of the order of one

2.3 Properties of He-II described using the two-fluid model

Momentum of heat flow: results plotted as $$\frac{FA}{\beta' \dot{Q}^2} = \frac{\varrho_n}{\varrho_s \varrho} \frac{1}{T^2 S^2}$$





Pyotr Leonidovich Kapitsa (1894 – 1984)

- results are independent of geometry
- because of $\varrho_n v_n + \varrho_s v_s = 0$ ⟶ rise at low and high T
- line: two-fluid model (without free parameter)

Two-Fluid Hydrodynamics

density
$$\varrho = \varrho_{\mathrm{n}} + \varrho_{\mathrm{s}} \quad (1)$$

mass flow
$$\boldsymbol{j} = \varrho_{\mathrm{n}}\boldsymbol{v}_{\mathrm{n}} + \varrho_{\mathrm{s}}\boldsymbol{v}_{\mathrm{s}} \quad (2)$$

mass conservation
continuity eqn.
$$\frac{\partial \varrho}{\partial t} = -\mathrm{div}\,\boldsymbol{j} \quad (3)$$

ideal fluid
$$\frac{\partial \boldsymbol{j}}{\partial t} = -\mathrm{grad}\,p \quad (4)$$

entropy conservation
$$\frac{\partial(\varrho S)}{\partial t} = -\mathrm{div}(\varrho S \boldsymbol{v}_{\mathrm{n}}) \quad (5)$$

an equation of motion for
superfluid component
$$\frac{\partial \boldsymbol{v}_{\mathrm{s}}}{\partial t} = S\,\mathrm{grad}\,T - \frac{1}{\varrho}\,\mathrm{grad}\,p \quad (6)$$

2.3 Properties of He-II described using the two-fluid model

d) Sound propagation (precision test of two-fluid model)

differentiation of (3) in respect to time and insert in (4)

$$\frac{\partial^2 \varrho}{\partial t^2} = \nabla^2 p \qquad (*)$$

$$\frac{\partial \rho}{\partial t} = -\mathrm{div}\,\vec{j} \qquad (3)$$

$$\frac{\partial^2 \rho}{\partial t^2} = -\mathrm{div}\left(\frac{\partial \vec{j}}{\partial t}\right)$$

$$\frac{\partial \vec{j}}{\partial t} = -\mathrm{grad}\, p \qquad (4)$$

$$\frac{\partial^2 \rho}{\partial t^2} = -\mathrm{div}(-\mathrm{grad}\, p)$$

$$\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p$$

eliminate $\boldsymbol{v}_\mathrm{s}$ and $\boldsymbol{v}_\mathrm{n}$ in (5) and (6) with (2)

since not observable

neglect terms of 2nd order

$$\frac{\partial^2 S}{\partial t^2} = \frac{\varrho_\mathrm{s}\, S^2}{\varrho_\mathrm{n}} \nabla^2 T \qquad (**)$$

with $(*)$ and $(**)$ one can fully describe the sound propagation in He-II
(under the assumption we made)

2.3 Properties of He-II described using the two-fluid model

we have 2 equations, but 4 variables $(\varrho,\ S,\ p,\ T)$ however, only 2 independent variables

We choose ϱ, S as independent and express p, T with ϱ and S (for small changes)

$$\left.\begin{aligned} \delta p &= \left(\frac{\partial p}{\partial \varrho}\right)_S \delta\varrho + \left(\frac{\partial p}{\partial S}\right)_\varrho \delta S\,, \\ \delta T &= \left(\frac{\partial T}{\partial \varrho}\right)_S \delta\varrho + \left(\frac{\partial T}{\partial S}\right)_\varrho \delta S \end{aligned}\right\}$$

insert in $(*)$ and $(**)$

$$\frac{\partial^2 \varrho}{\partial t^2} = \left(\frac{\partial p}{\partial \varrho}\right)_S \nabla^2\varrho + \left(\frac{\partial p}{\partial S}\right)_\varrho \nabla^2 S$$

$$\frac{\partial^2 S}{\partial t^2} = \frac{\varrho_s}{\varrho_n} S^2 \left[\left(\frac{\partial T}{\partial \varrho}\right)_S \nabla^2\varrho + \left(\frac{\partial T}{\partial S}\right)_\varrho \nabla^2 S\right]$$

2 partial differential equations of 2nd order