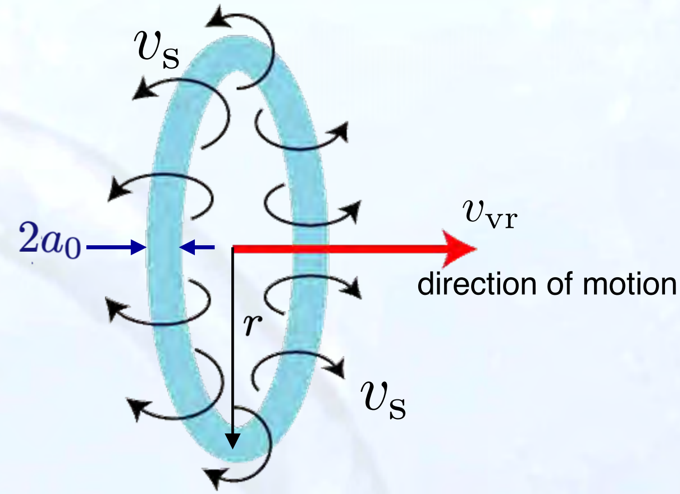
2.7 Motion of Ions in He-II

vortex rings

kinetic energy of vortex ring: He-II $\varrho \rightarrow \varrho_\mathrm{s}$

$$E_\mathrm{vr} = \int \frac{1}{2}\varrho_\mathrm{s} v_\mathrm{s}^2 \mathrm{d}V = \frac{1}{2}\varrho_\mathrm{s}\kappa^2 r \left[\ln\left(\frac{8r}{a_0}\right) - \frac{7}{4}\right] \quad \propto r$$

momentum of vortex ring $p_\mathrm{vr} = \pi \varrho_\mathrm{s} \kappa r^2$



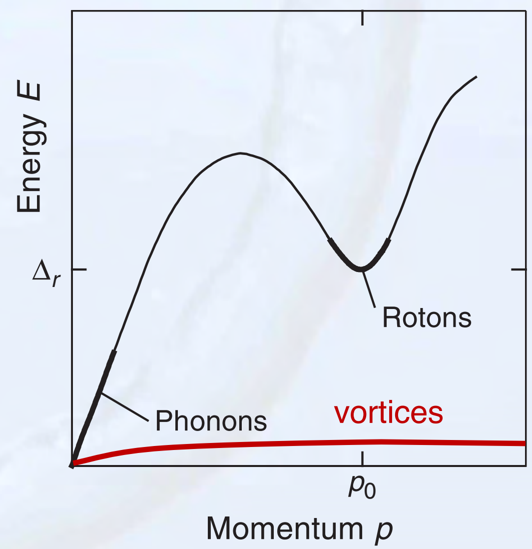
$$\Rightarrow \quad v_\mathrm{vr} = \frac{\partial E}{\partial p_\mathrm{vr}} = \frac{\kappa}{4\pi r}\left[\ln\left(\frac{8r}{a_0}\right) - \frac{1}{4}\right]$$

$\Rightarrow$ $p_\mathrm{vr} \propto r^2 \propto E_\mathrm{vr}^2$ and $v_\mathrm{vr} \propto 1/E$

as observed

$\Rightarrow$ $\boxed{E_\mathrm{vr} \propto \sqrt{p_\mathrm{vr}}}$

dispersion of vortex ring



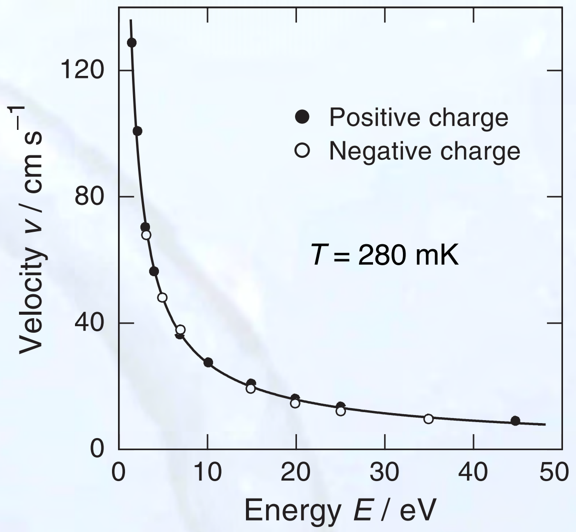
2.7 Motion of Ions in He-II

Explanation of the experiment by Rayfield and Reif

- generation of vortex rings
- ions are captured by vortex ring
- field increases kinetic energy of vortex ring

$$v_{\mathrm{vr}} \propto \frac{1}{r} \propto \frac{1}{E_{\mathrm{vr}}}$$

- theory line with $a_0 = 1.2\ \text{Å}$

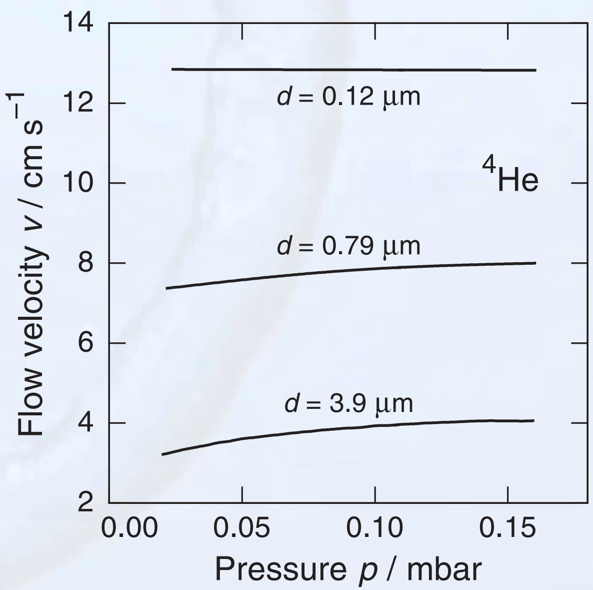


let's revisit the flow experiments through capillaries

because of $E_{\mathrm{vr}} \propto \sqrt{p_{\mathrm{vr}}}$, largest possible vortex is has minimal critical velocity

for capillary with diameter d

$$v_{\mathrm{c,vr}} = \frac{\hbar}{m_4 d}\left[\ln\left(\frac{4d}{a_0}\right) - \frac{1}{4}\right] \propto \frac{1}{d}$$



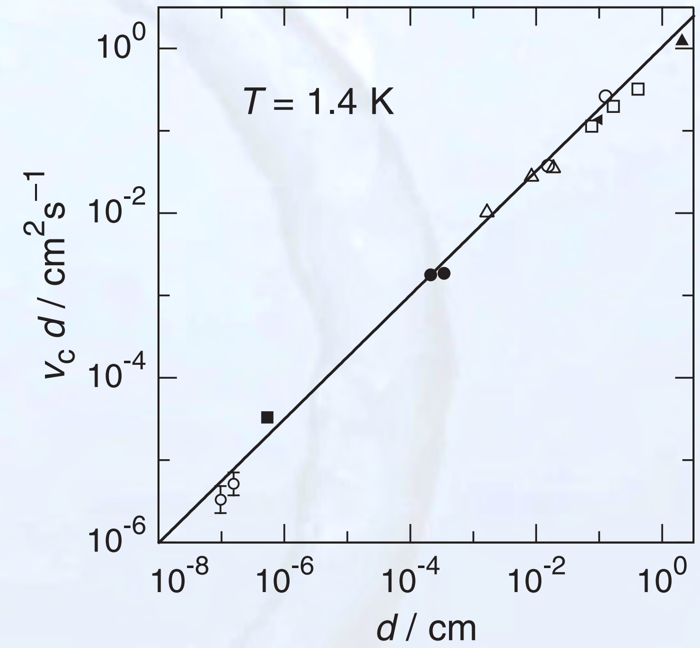
➡ qualitative agreement with flow experiments in capillaries

2.7 Motion of Ions in He-II

flow experiments to determine the critical velocity

how does the critical velocity depend on d ?

- potted is: $v_{\mathrm{c}} d$ vs d
- critical velocity $v_{\mathrm{c}} \propto d^{-1/4}$
- expected $v_{\mathrm{c}} \propto d^{-1}$
- reason is unknown



2.7 Critical Behaviour of He-II at T_λ

Properties near T_c are determined by quantities that go to zero like the order parameter and quantities that diverge like susceptibilities

Landau theory of continuous phase transitions (1937, 1965)

- idea: expansion of free energy in T in terms of the order parameter
- near T_c one should find the following laws with the reduced temperature $t = (T - T_c)/T_c$

Quantity	Power Law	Critical Exponent
specific heat	$C_V \propto \lvert t\rvert^{\alpha}$	$\alpha = 0$
order parameter	$\Phi \propto \lvert t\rvert^{\beta}$	$\beta = 1/2$
susceptibility	$\chi \propto \lvert t\rvert^{-\gamma}$	$\gamma = 1$
correlation length	$\xi \propto \lvert t\rvert^{-\nu}$	$\nu = 1/2$

Landau type theories: – van der Waals theory for liquid – gas transition
– Curie-Weiss theory of ferromagnetism
– Ginzburg-Landau theory of superconductivity

2.7 Critical Behaviour of He-II at T_λ

Problem: fluctuations are not included, but they are increasingly important towards T_c

→ every Landau-type theory breaks down near T_c

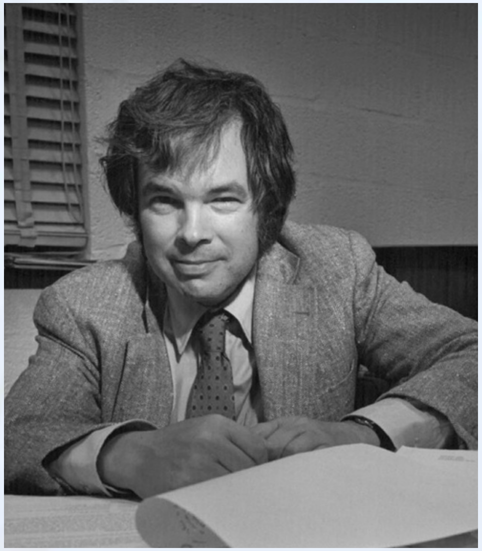
Ginzburg criterion

The condition under which a Landau-type theory holds is that fluctuations
of the order parameter are small in comparison of the mean value of the order parameter

for He-II: coherence length is very small ⟶ Ginzburg criterion is ”always” violated

Renormalization group

Despite of the short-comings of the Landau universal theory of phase transitions, it was realized that it is possible to assign different physical systems to universality classes, characterized by a set of critical exponents



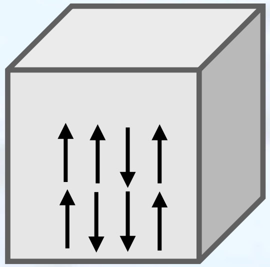
Kenneth G. Wilson

The larger framework is: renormalization group and quantum field theory

different classes are defined by: dimension of system d,
degrees of freedom of order parameter n,
interaction length compared to coherence length

2.7 Critical Behaviour of He-II at T_λ

a few examples: Ising 3 D

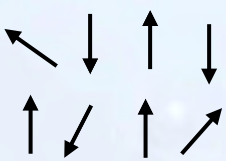


$d = 3$
$n = 1$

in this universality class liquid-solid transition fall as well

Heisenberg 2 D

$d = 2$
$n = 3$



at each lattice point each spin can point in 3 direction

x–y 3 D

$d = 3$
$n = 2$

He-II
superconductors

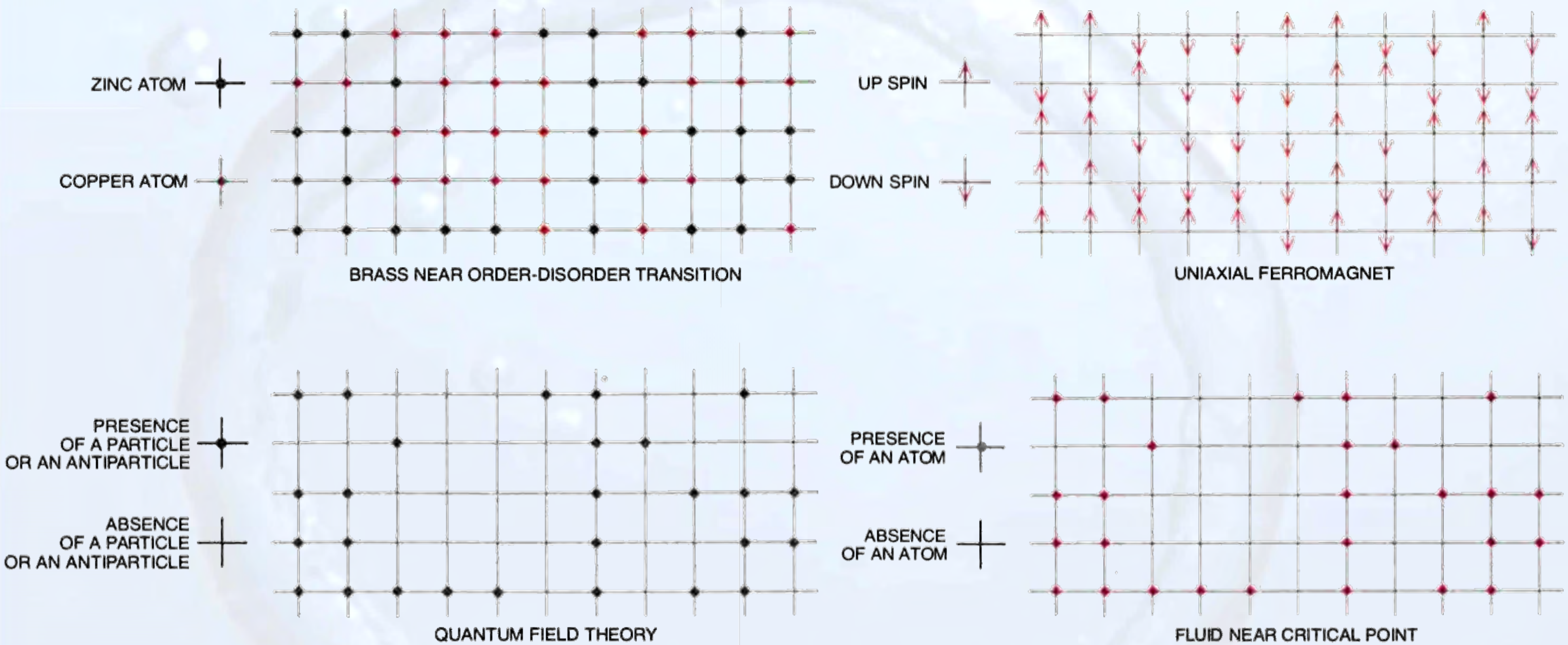
magnitude and phase of wave function

each universality class is described by a set of critical exponents and are connected by sum rules like $\alpha + 2\beta + \gamma = 2$

2.7 Critical Behaviour of He-II at T_λ

UNIVERSALITY CLASS		THEORETICAL MODEL	PHYSICAL SYSTEM	ORDER PARAMETER
$d = 2$	$n = 1$	Ising model in two dimensions	Adsorbed films	Surface density
	$n = 2$	XY model in two dimensions	Helium-4 films	Amplitude of superfluid phase
	$n = 3$	Heisenberg model in two dimensions		Magnetization
$d > 2$	$n = \infty$	"Spherical" model	None	
$d = 3$	$n = 0$	Self-avoiding random walk	Conformation of long-chain polymers	Density of chain ends
	$n = 1$	Ising model in three dimensions	Uniaxial ferromagnet	Magnetization
			Fluid near a critical point	Density difference between phases
			Mixture of liquids near consolute point	Concentration difference
			Alloy near order-disorder transition	Concentration difference
	$n = 2$	XY model in three dimensions	Planar ferromagnet	Magnetization
			Helium 4 near superfluid transition	Amplitude of superfluid phase
	$n = 3$	Heisenberg model in three dimensions	Isotropic ferromagnet	Magnetization
$d \leq 4$	$n = -2$		None	
	$n = 32$	Quantum chromodynamics	Quarks bound in protons, neutrons, etc.	

2.7 Critical Behaviour of He-II at T_λ



2.7 Critical Behaviour of He-II at T_λ

critical exponents expected for X-Y 3D model:

$\alpha = -0.0146(8)$

$\beta = 0.3485(2)$

$\gamma = 1.3177(5)$

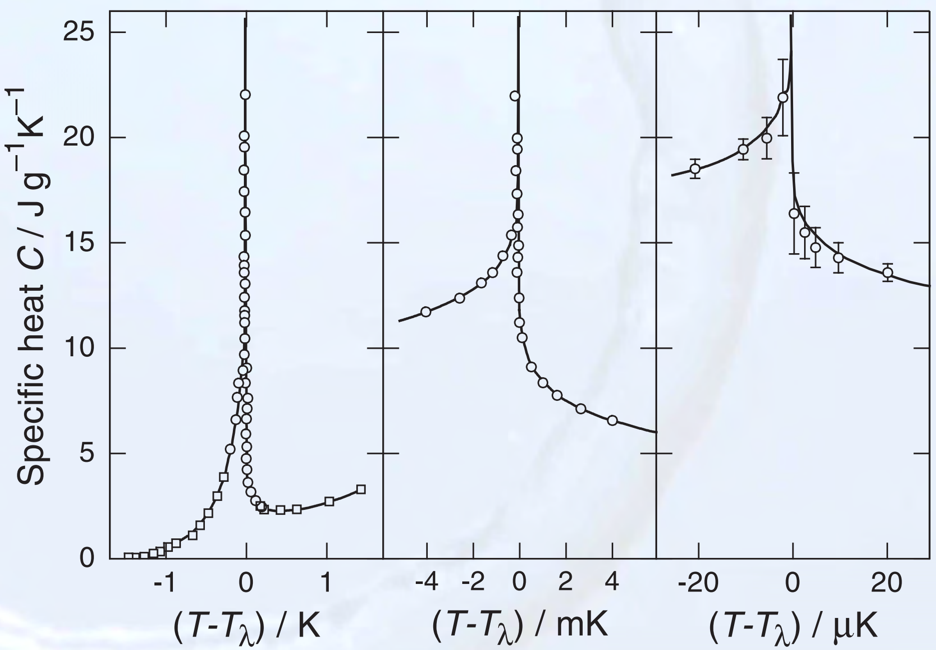
$\delta = 4.780(2)$

$\nu = 0.67155(27)$

$\eta = 0.0380(4)$

Experiments near T_λ

a) specific heat



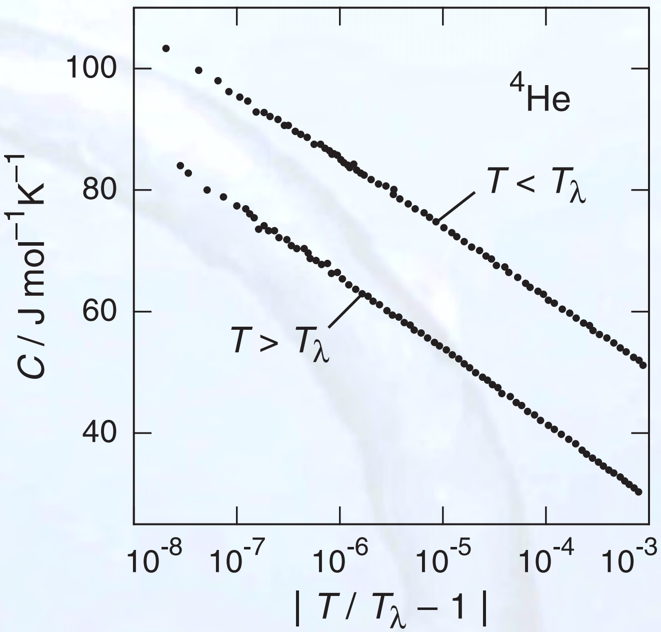
scale going from K to μK

2.7 Critical Behaviour of He-II at T_λ

power law in the vicinity of T_λ ?

data plotted a C_V vs $\log t = \log|T/T_\lambda - 1|$

data can be approximated by $C_V \propto \log t$



logarithmic divergences?

comparison with RGT

expected scaling for He-II

$$C = B + A\,\frac{t^{-\alpha}}{\alpha}\left(1 - D\sqrt{t}\right)$$ $A,\ B$ and D are constants

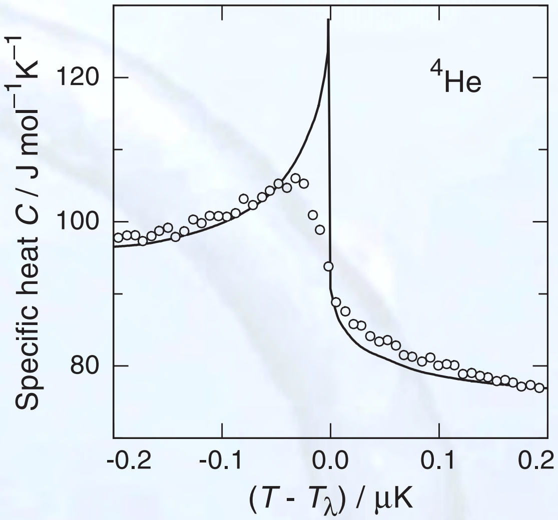
with critical exponent expected $\alpha = -0.146(8)$

➡ expansion in α $t^{-\alpha} = \mathrm{e}^{-\alpha \ln t} \approx 1 - \alpha \ln t$ expansion justified because of small α

experimental result $\alpha \approx -0.013 \pm 0.003$

2.7 Critical Behaviour of He-II at T_λ

Higher precision experiments near T_λ are needed



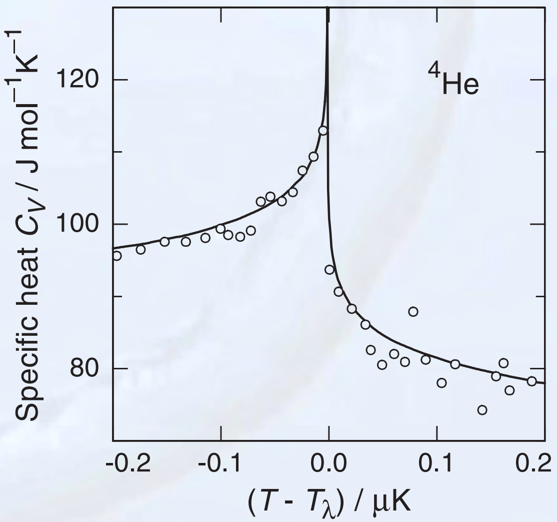
measurement on earth

Problems:

gravitation ⟶ level height dependence

walls of vessel ⟶ first layer solid and healing length diverges with diverges near T_λ with $\xi = \xi_0 \, t^{-\nu}$ with $\nu = 0.67155(27)$

measurement on space shuttle

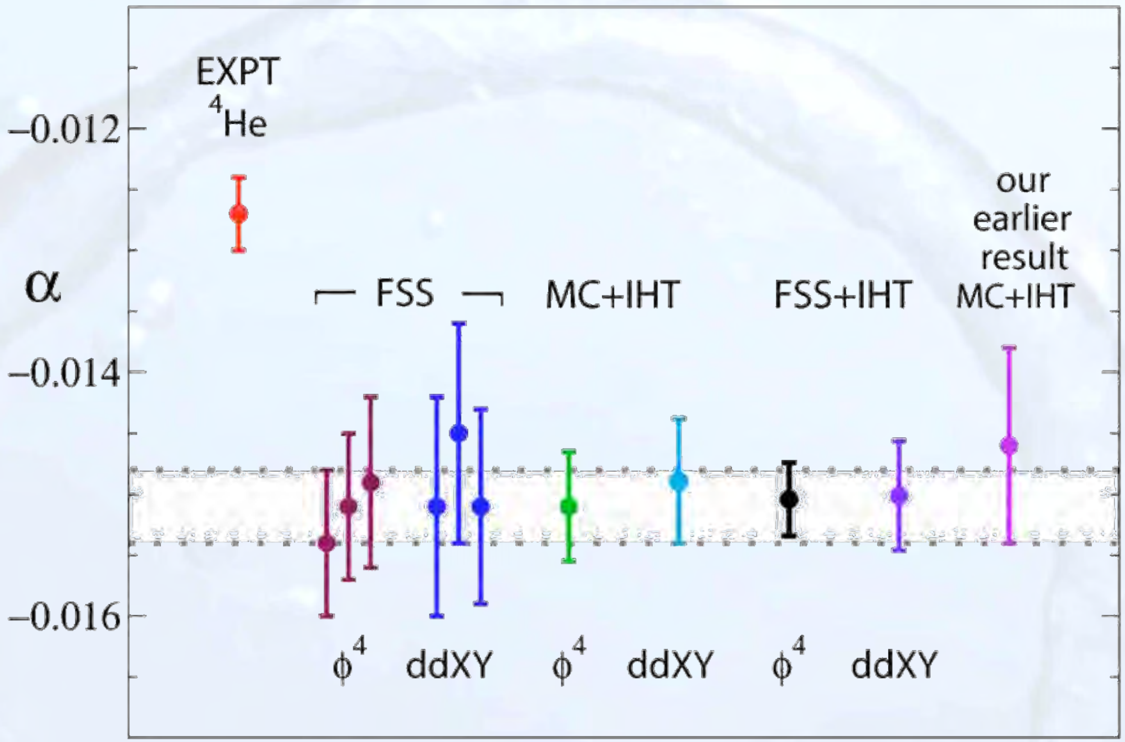


Problems:
cosmic rays ⟶ time varying background (heating of thermometer)

Data shown, after sophisticated analysis

⟶ still somewhat noisy

2.7 Critical Behaviour of He-II at T_λ



comparison between space shuttle data and different calculations of α

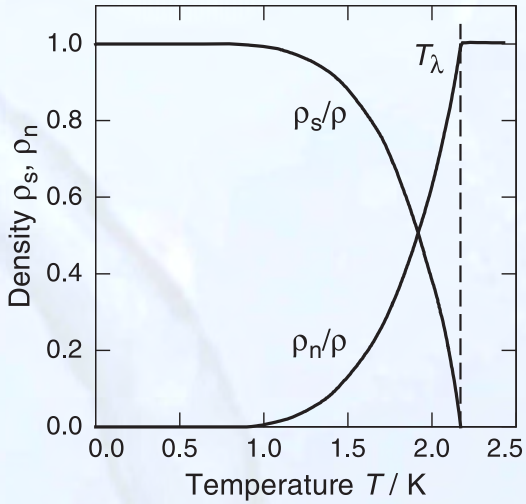
→ discrepancy between data and theory outside error bars: reason unknown

2.7 Critical Behaviour of He-II at T_λ

b) Order parameter

amplitude of wave function

$$\psi(\boldsymbol{r}) = \psi_0 \, \mathrm{e}^{\mathrm{i}\varphi(\boldsymbol{r})} \longrightarrow \Psi_0 = \sqrt{\varrho_s}$$



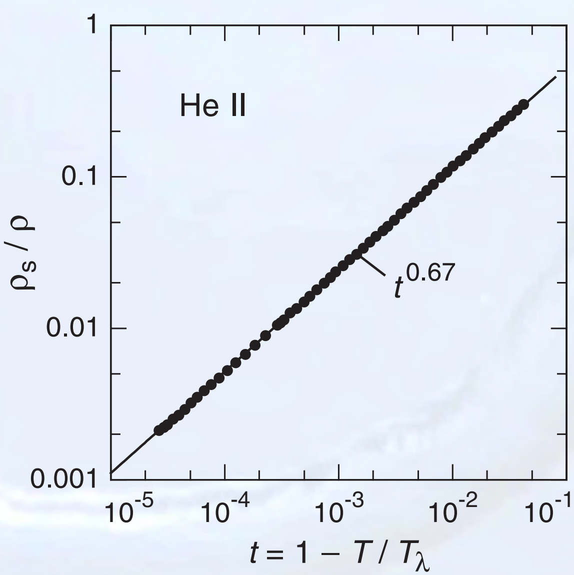
expected:

$$\varrho_s = t^{2\beta} \quad \text{with} \quad \beta = 0.3485(2)$$

determined with second sound

$$\varrho_s = t^{0.67}$$

➡ excellent agreement



2.7 Critical Behaviour of He-II at T_λ

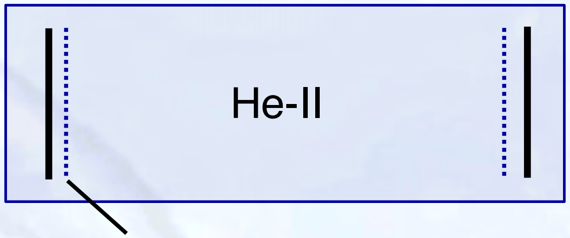
c) Healing length

again, second sound measurements
and measurements on thin films

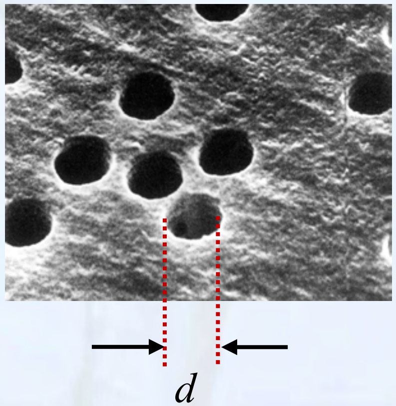
expected:

$\xi = \xi_0 \, t^{-\nu}$ with $\nu = 0.67155(27)$

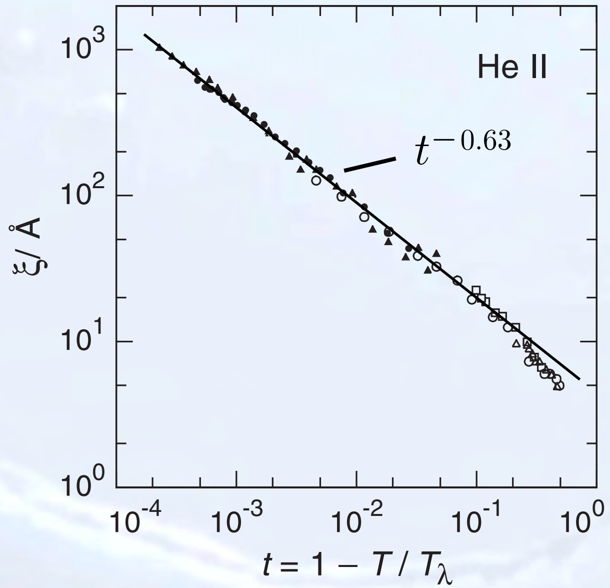
Helmholtz resonator



Nuclepore filters



$$\xi_0 = 2.8 \pm 0.5\,\text{Å}$$
$$\nu \;=\; 0.63$$



→ excellent agreement

second sound vanishes
for $\xi > d$