2.3 Properties of He-II described using the two-fluid model

Momentum of heat flow

Heat flow in He-II $\longrightarrow$ momentum flow $\varrho\,\boldsymbol{v}\cdot\boldsymbol{v}$

momentum flow / volume

resulting pressure acting on a heat source

$$p = \varrho_\mathrm{n} v_\mathrm{n}^2 + \varrho_\mathrm{s} v_\mathrm{s}^2 \qquad (*)$$

no net mass transport (closed vessel)

$$\varrho_\mathrm{n} v_\mathrm{n} + \varrho_\mathrm{s} v_\mathrm{s} = 0 \longrightarrow v_\mathrm{s} = -\frac{\varrho_\mathrm{n}}{\varrho_\mathrm{s}} v_\mathrm{n}$$

insert in $(*)$ $\longrightarrow$ $p = \dfrac{\varrho_\mathrm{n}\varrho}{\varrho_\mathrm{s}} v_\mathrm{n}^2$

$$\frac{\dot V_\mathrm{n}}{A}$$

with heat flow / per area $\dfrac{\dot Q}{A} = \varrho S T v_\mathrm{n} \longrightarrow v_\mathrm{n} = \dfrac{\dot Q}{A\varrho S T}$

$$\boxed{p = \frac{\varrho_\mathrm{n}}{\varrho_\mathrm{s}\varrho}\left(\frac{\dot Q}{AST}\right)^2}$$

pressure associated with uni-directional heat flow

$$p = \varrho_n v_n^2 + \varrho_s v_s^2$$

$$v_s = -\frac{\varrho_n}{\varrho_s} v_n$$

$$= \varrho_n v_n^2 + \varrho_s \left(-\frac{\varrho_n}{\varrho_s} v_n\right)^2$$

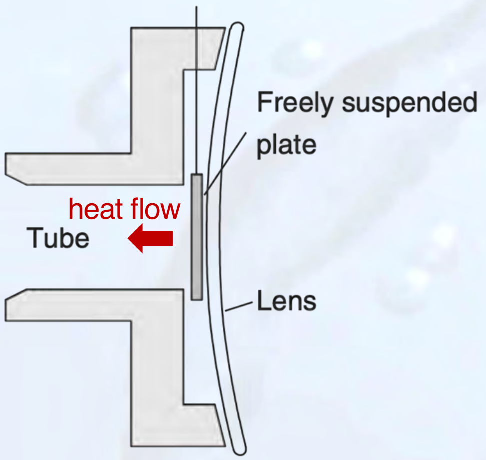
$$= \varrho_n v_n^2 \left(1 + \frac{\varrho_n}{\varrho_s}\right)$$

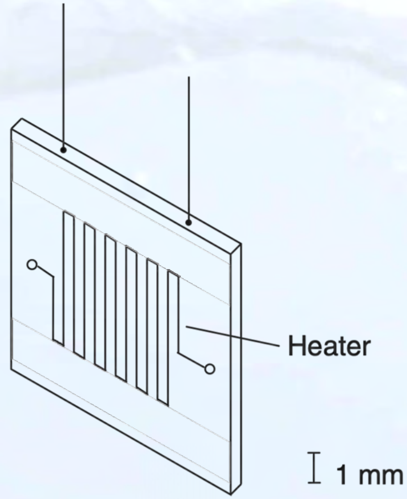
$$= \varrho_n v_n^2 \left(\frac{\varrho_s + \varrho_n}{\varrho_s}\right)$$

$$= \frac{\varrho_n \varrho}{\varrho_s} v_n^2$$

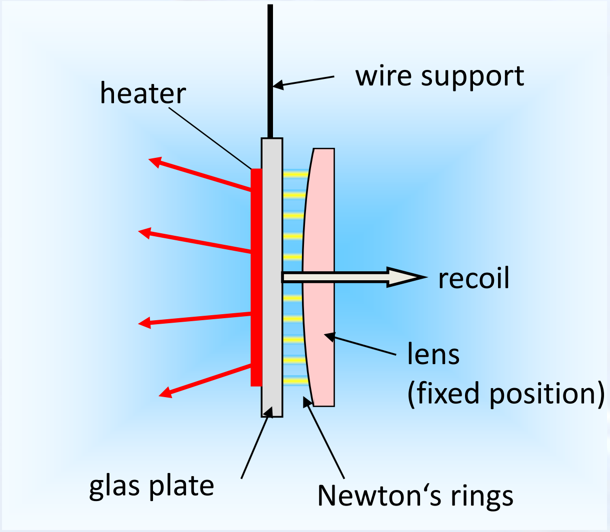
2.3 Properties of He-II described using the two-fluid model

Momentum of heat flow: Measurement

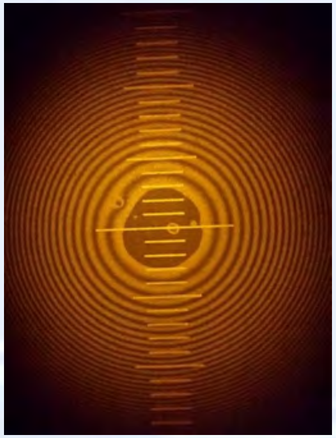




change of distance between glass plate and lens measured by Newton rings $\longrightarrow$ force



Newton rings



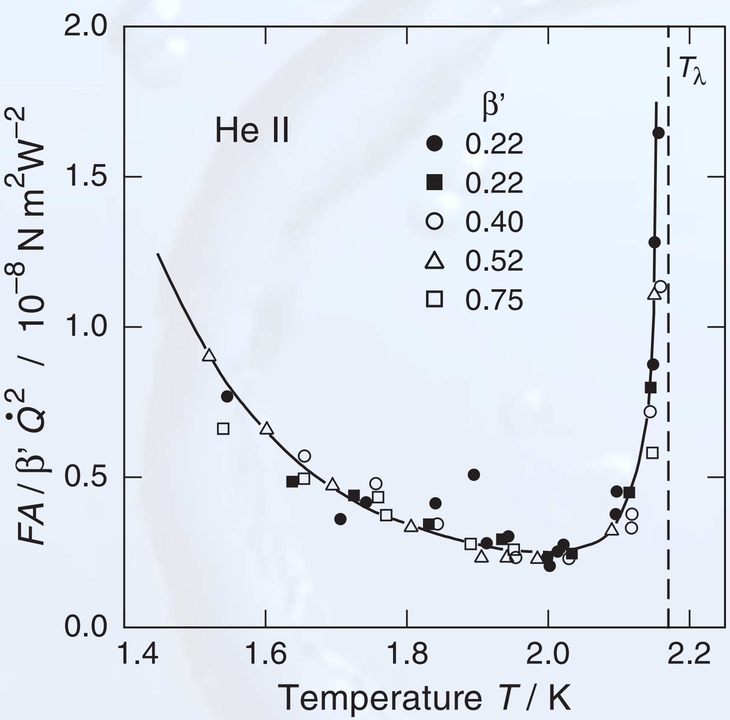
Expected force

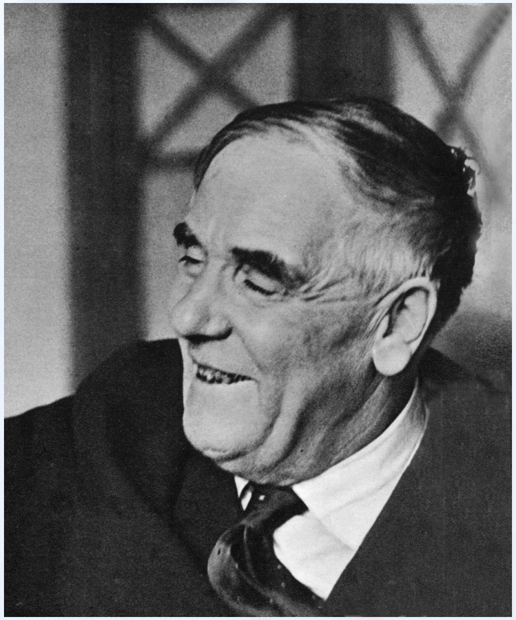
$$F = pA = \beta' \frac{\varrho_\mathrm{n}}{\varrho_\mathrm{s} \varrho A} \left(\frac{\dot{Q}}{ST} \right)^2$$

geometry dependent factor of the order of one

2.3 Properties of He-II described using the two-fluid model

Momentum of heat flow: results plotted as $$\frac{FA}{\beta'\dot{Q}^2} = \frac{\varrho_n}{\varrho_s \varrho}\frac{1}{T^2 S^2}$$





Pyotr Leonidovich Kapitsa (1894 – 1984)

- results are independent of geometry
- because of $\varrho_n v_n + \varrho_s v_s = 0$ ⟶ rise at low and high T
- line: two-fluid model (without free parameter)

Two-Fluid Hydrodynamics

density $$\varrho = \varrho_{\mathrm{n}} + \varrho_{\mathrm{s}} \tag{1}$$

mass flow $$\boldsymbol{j} = \varrho_{\mathrm{n}} \boldsymbol{v}_{\mathrm{n}} + \varrho_{\mathrm{s}} \boldsymbol{v}_{\mathrm{s}} \tag{2}$$

mass conservation
continuity eqn.
$$\frac{\partial \varrho}{\partial t} = -\mathrm{div}\, \boldsymbol{j} \tag{3}$$

ideal fluid $$\frac{\partial \boldsymbol{j}}{\partial t} = -\mathrm{grad}\, p \tag{4}$$

entropy conservation $$\frac{\partial (\varrho S)}{\partial t} = -\mathrm{div}(\varrho S \boldsymbol{v}_{\mathrm{n}}) \tag{5}$$

an equation of motion for
superfluid component
$$\frac{\partial \boldsymbol{v}_{\mathrm{s}}}{\partial t} = S\, \mathrm{grad}\, T - \frac{1}{\varrho}\, \mathrm{grad}\, p \tag{6}$$

2.3 Properties of He-II described using the two-fluid model

d) Sound propagation (precision test of two-fluid model)

differentiation of (3) in respect to time and insert in (4)

$$\frac{\partial^2 \varrho}{\partial t^2} = \nabla^2 p \qquad (*)$$

eliminate $\boldsymbol{v}_\mathrm{s}$ and $\boldsymbol{v}_\mathrm{n}$ in (5) and (6) with (2)

since not observable

neglect terms of 2nd order

$$\frac{\partial^2 S}{\partial t^2} = \frac{\varrho_\mathrm{s}\, S^2}{\varrho_\mathrm{n}} \nabla^2 T \qquad (**)$$

with $(*)$ and $(**)$ one can fully describe the sound propagation in He-II
(under the assumption we made)

$$\frac{\partial \varrho}{\partial t} = -\operatorname{div} \vec{j} \qquad (3)$$

$$\frac{\partial^2 \varrho}{\partial t^2} = -\operatorname{div}\left(\frac{\partial \vec{j}}{\partial t}\right)$$

$$\frac{\partial \vec{j}}{\partial t} = -\operatorname{grad} p \qquad (4)$$

$$\frac{\partial^2 \varrho}{\partial t^2} = -\operatorname{div}(-\operatorname{grad} p)$$

$$\frac{\partial^2 \varrho}{\partial t^2} = \nabla^2 p$$

2.3 Properties of He-II described using the two-fluid model

we have 2 equations, but 4 variables $(\varrho,\ S,\ p,\ T)$ however, only 2 independent variables

We choose ϱ, S as independent and express p, T with ϱ and S (for small changes)

$$\left.\begin{aligned}\delta p &= \left(\frac{\partial p}{\partial \varrho}\right)_S \delta\varrho + \left(\frac{\partial p}{\partial S}\right)_\varrho \delta S\,, \\ \delta T &= \left(\frac{\partial T}{\partial \varrho}\right)_S \delta\varrho + \left(\frac{\partial T}{\partial S}\right)_\varrho \delta S\end{aligned}\right\}$$

insert in $(*)$ and $(**)$

$$\boxed{\begin{aligned}\frac{\partial^2 \varrho}{\partial t^2} &= \left(\frac{\partial p}{\partial \varrho}\right)_S \nabla^2\varrho + \left(\frac{\partial p}{\partial S}\right)_\varrho \nabla^2 S \\ \frac{\partial^2 S}{\partial t^2} &= \frac{\varrho_\mathrm{s}}{\varrho_\mathrm{n}} S^2 \left[\left(\frac{\partial T}{\partial \varrho}\right)_S \nabla^2\varrho + \left(\frac{\partial T}{\partial S}\right)_\varrho \nabla^2 S\right]\end{aligned}}$$

2 partial differential equations of 2nd order

2.3 Properties of He-II described using the two-fluid model

Ansatz:
$$\varrho = \varrho_0 + \varrho' \, \mathrm{e}^{\mathrm{i}\omega(t-x/v)} \,,$$

$$S = S_0 + S' \, \mathrm{e}^{\mathrm{i}\omega(t-x/v)}$$

frequency of wave

velocity in x direction

Insertion and differentiation leads to 2 linear equations in ϱ' and S'

$$\left[\left(\frac{v}{v_1}\right)^2 - 1\right] \varrho' + \left(\frac{\partial p}{\partial S}\right)_\varrho \left(\frac{\partial \varrho}{\partial p}\right)_S S' = 0 \,, \qquad \text{(i)}$$

$$\left(\frac{\partial T}{\partial \varrho}\right)_S \left(\frac{\partial S}{\partial T}\right)_\varrho \varrho' + \left[\left(\frac{v}{v_2}\right)^2 - 1\right] S' = 0 \qquad \text{(ii)}$$

with

$$v_1^2 = \left(\frac{\partial p}{\partial \varrho}\right)_S \qquad \text{and} \qquad v_2^2 = \frac{\varrho_\mathrm{s}}{\varrho_\mathrm{n}} S^2 \left(\frac{\partial T}{\partial S}\right)_\varrho$$

2.3 Properties of He-II described using the two-fluid model

the constrains equation for the coefficients is

$$\left[\left(\frac{v}{v_1}\right)^2-1\right]\left[\left(\frac{v}{v_2}\right)^2-1\right]=\underbrace{\left(\frac{\partial p}{\partial S}\right)_\varrho\left(\frac{\partial \varrho}{\partial p}\right)_S\left(\frac{\partial T}{\partial \varrho}\right)_S\left(\frac{\partial S}{\partial T}\right)_\varrho}$$

here standard thermodynamic relations are used

$$\boxed{\left[\left(\frac{v}{v_1}\right)^2-1\right]\left[\left(\frac{v}{v_2}\right)^2-1\right]=\frac{C_p-C_V}{C_p}}$$

for liquid helium $C_p \approx C_V$

$$\left[\left(\frac{v}{v_1}\right)^2-1\right]\left[\left(\frac{v}{v_2}\right)^2-1\right]\approx 0 \qquad \text{(iii)}$$

interpretation: two wave v_1, v_2 weakly coupled via $\frac{C_p-C_V}{C_p}$

2.3 Properties of He-II described using the two-fluid model

(i) First sound

with (i) and (iii)

$$v = v_1 = \sqrt{\left(\frac{\partial p}{\partial \varrho}\right)_S}$$

$$\left(\frac{v}{v_1}\right)^2 - 1 = 0$$

$$\varrho' \neq 0 \quad S' = 0$$

$$\longrightarrow \quad \mathrm{grad}\,T = 0$$

as usual for ordinary (first) sound

insert (4) into (6)

$$\varrho_\mathrm{n} \frac{\partial}{\partial t} \underbrace{(\boldsymbol{v}_\mathrm{n} - \boldsymbol{v}_\mathrm{s})} = \varrho S \,\mathrm{grad}\, T = 0$$

$\boldsymbol{v}_\mathrm{n} = \boldsymbol{v}_\mathrm{s}$ $\Rightarrow$ superfluid and normalfluid component are in phase

(4) in (6)

$$\frac{\partial \vec{v}_s}{\partial t} = S\,\mathrm{grad}\,T + \frac{1}{\varrho}\frac{\partial \vec{j}}{\partial t}$$

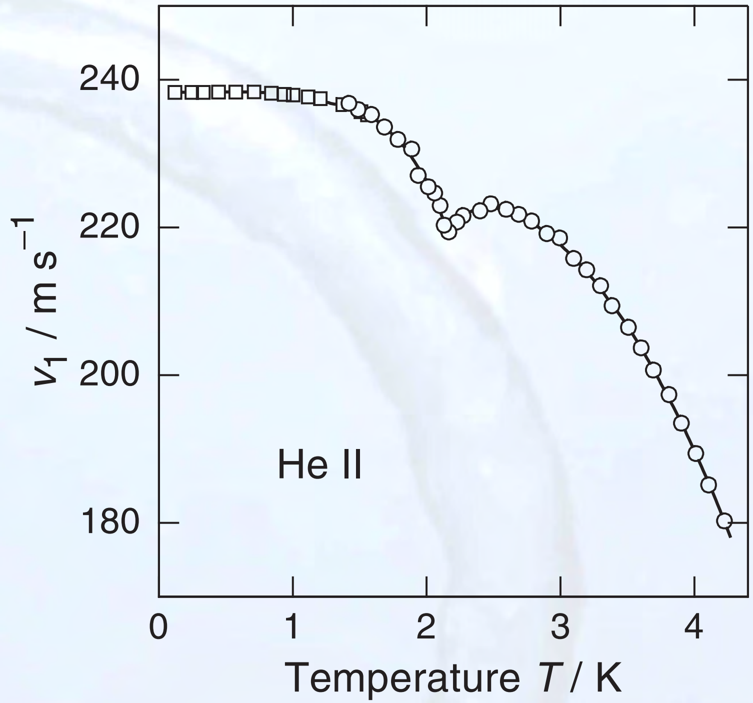
insert (2) $\times \varrho$

$$\curvearrowright \; \varrho\left(\frac{\partial \vec{v}_s}{\partial t}\right) = \varrho S\,\mathrm{grad}\,T + \varrho_n \frac{\partial \vec{v}_n}{\partial t} + \varrho_s \frac{\partial \vec{v}_s}{\partial t}$$

$$\curvearrowright \; \varrho_n \frac{\partial}{\partial t}\left(\vec{v}_s - \vec{v}_n\right) = \varrho S\,\mathrm{grad}\,T = 0$$

2.3 Properties of He-II described using the two-fluid model

(i) First sound



- for $T \to 0$: $v_1 \approx 238\,\mathrm{m\,s^{-1}}$.
  ⟶ only density variation ⟶ almost ordinary sound
- for $T \to T_\lambda$: corrections become important

2.3 Properties of He-II described using the two-fluid model

(ii) Second sound

with (ii) and (iii) we find

$$v = v_2 = \sqrt{\frac{\varrho_\mathrm{s}}{\varrho_\mathrm{n}} S^2 \left(\frac{\partial T}{\partial S}\right)_\varrho}$$

$$\left(\frac{v}{v_2}\right)^2 - 1 = 0 \qquad S' \neq 0 \ , \quad \varrho' = 0$$

$$\longrightarrow \quad \mathrm{grad}\, p = 0$$

with (4)

$$\frac{\partial \boldsymbol{j}}{\partial t} = -\mathrm{grad}\, p \stackrel{!}{=} 0 \quad \Rightarrow \quad \frac{\partial \varrho_\mathrm{n} \boldsymbol{v}_\mathrm{n}}{\partial t} + \frac{\partial \varrho_\mathrm{s} \boldsymbol{v}_\mathrm{s}}{\partial t} = 0$$

$$\varrho_\mathrm{n} \boldsymbol{v}_\mathrm{n} + \varrho_\mathrm{s} \boldsymbol{v}_\mathrm{s} = 0$$

no mass flow in closed vessel

$\varrho_\mathrm{n} \uparrow \ , \quad \varrho_\mathrm{s} \downarrow$ counter flow and no density variation

$\Rightarrow$ temperature wave

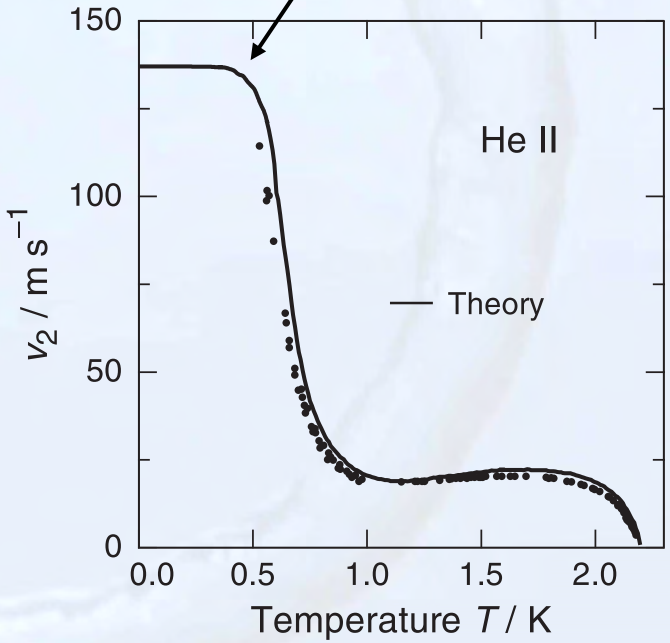
2.3 Properties of He-II described using the two-fluid model

$$\boxed{v_2 = \sqrt{\frac{\varrho_s}{\varrho_n} S^2 \left(\frac{\partial T}{\partial S}\right)_\varrho} = \sqrt{\frac{\varrho_s}{\varrho_n} \frac{T\,S^2}{C_p}}}$$

density variation in phonon gas

possibility to determine ϱ_s / ϱ_n

for $T \rightarrow 0$ second sound difficult determine since $\varrho_n \rightarrow 0$



ultra-low temperatures:

excitations at $T \rightarrow 0$ are only longitudinal phonons

Landau

Debye model

$A = 2\pi^2 k_B^4 / (45\hbar^3 v_1^3 \varrho)$

$C_p = 3AT^3$

$S = AT^3$

in addition

$\varrho_s \approx \varrho$

$\varrho_n = A\varrho T^4 / v_1^2$

for $T \rightarrow 0$:

$v_2 \rightarrow v_1/\sqrt{3} \approx 137\,\mathrm{m\,s^{-1}}$