Recitation 5 – Moderns Aspects of Nuclear Physics – SS22

Discussion on Fr., July 1, 2022, 11:15

5.1 Gravitational binding energy of a neutron star (1 Point)

Assume a neutron star with radius $r_{ns} = 10\text{km}$ and mass $m_{ns} = 1.5 \cdot M_{\text{sun}}$, with M_{sun} being the mass of the sun.

a) Calculate the gravitational energy of this neutron star assuming a uniform and homogeneous sphere.

b) Calculate the gravitational energy per nucleon and compare your result to the average nuclear binding energy.

c) Compare the density of a neutron star to normal nuclear matter of $\rho_{\text{nuclear}} \approx 0.17\text{nucleons/fm}^3$.

Hint: For this problem, use SI units as an exception.

Solution:

a) $\rho = \text{constant}$, $\rho = m_{ns}/V = m_{ns}/(4/3 \cdot \pi r_{ns}^3)$

The change in potential energy when bringing an infinitely thin layer of mass from infinity to an existing volume of mass with radius r is:

$$dW = -G\rho \cdot 4/3\pi r^3 \cdot r^2 \rho \cdot 4\pi dr = -G\frac{16\pi^2}{3} \cdot \rho^2 r^4 dr.$$

Integrating over $r = 0...r = r_{ns}$ gives

$$W = -G\frac{\rho^2 \cdot 16\pi^2}{15} \cdot r_{ns}^5$$

Inserting ρ from above results in

$$W = -G\frac{16\pi^2}{15} \cdot r_{ns}^5 \cdot \frac{9m_{ns}^2}{16\pi^2 \cdot r_{ns}^6}$$

$$W = -\frac{3}{5} \cdot G\frac{m_{ns}^2}{r_{ns}}$$

$G = 6.67 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}$, $M_{\text{sun}} = 2 \cdot 10^{30}\text{kg}$.

$W = \frac{3}{5} \cdot 6.67 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2} \cdot (3 \cdot 10^{30}\text{kg})^2/10^4\text{m} = 3.6 \cdot 10^{46}\text{J}$.

b) $1u = 1.66 \cdot 10^{-24}\text{g}$

$n_{\text{nucleons}} = 3 \cdot 10^{30}\text{kg}/1.66 \cdot 10^{-24}\text{g} = 1.8 \cdot 10^{57}$.

Gravitational energy per nucleon $= 3.6 \cdot 10^{46}\text{J}/1.8 \cdot 10^{57}$, 1 Joule $= 6.24 \cdot 10^{+12}\text{MeV}$.

Gravitational energy per nucleon $= 3.6/1.8 \cdot 6.24 \cdot 10\text{MeV} = 125$ MeV per nucleon, nuclear binding energy ≈ 8 MeV per nucleon.

c) $\rho_{ns} = 1.8 \cdot 10^{57}/(4/3 \cdot \pi r_{ns}^3) = \frac{1.8 \cdot 3}{4\pi} \frac{10^{57}}{10^3 \cdot 10^9 \cdot 10^{45}\text{fm}^3} \approx 0.5\text{nucleons/fm}^3 \approx 3 \cdot \rho_{\text{nuclear}}$.

5.2 Extension of the liquid drop model (1 Point)

The liquid drop model can be extended to apply it to neutron stars. For this purpose, a term B_G is added to the binding energy that describes the gravitational attraction for very large neutron numbers:

$$B_G = \frac{3}{5}\frac{G \cdot m_n^2}{r_0} A^{5/3},$$

with the gravitational constant

$$G = 6.7 \cdot 10^{-39} \frac{1}{(\text{GeV})^2},$$

the neutron radius $r_0 = 0.88\text{fm}$ and the mass $m_n = 0.9396\text{GeV}$ of the neutron.

Estimate the minimum number of neutrons N a neutron star must posses in order to be stable against strong decay.

Hint: Use the approximation $A = N$ and $Z = 0$ for very large values of N. Use the stability criterion that the binding energy is positive.

Solution:

With $A \approx N$ and $Z \approx 0$ only the volume term and the symmetry term remain. The binding energy must be positive in order for the neutron star to be stable against strong decay.

$B = b_v \cdot N - b_{\text{sym}} \cdot N + \frac{3}{5}\frac{GM^2}{r_0} A^{5/3} > 0.$

$$N > \left(\frac{5}{3}\frac{(b_{\text{sym}} - b_v) r_0}{GM^2}\right)^{3/2} = \left(\frac{5}{3} \cdot \frac{(23.2 - 15.85) \cdot 0.88}{6.7 \cdot 10^{-39} \cdot 197 \cdot 0.9396^2}\right)^{3/2} = 2.8 \cdot 10^{55}.$$

N.B. $B_G = 7.945 \cdot 10^{-37} N^{5/3}$.

5.3 Chandrasekhar mass limit from dimensional analysis (1 Point)

Electron degeneracy creates a barrier to the gravitational collapse of dying stars and is responsible for the formation of white dwarfs. In the ultra-relativistic limit, the total energy E_t of the electrons is given by $E_t = N \cdot \frac{3}{5} E_F = N \cdot \frac{3}{5}\left(3\pi^2 \cdot \frac{N}{V}\right)^{1/3}$, with N the number of electrons, V the volume of the white dwarf, and E_F the Fermi energy of the electrons. The Fermi pressure is derived by $P_e = -\frac{\partial E}{\partial V} \sim \left(\frac{N}{V}\right)^{4/3}$.

In the following, we perform a dimensional analysis, e.g. we neglect constant factors such as $3/5, 4\pi, 3\pi^2$, *etc.*

a) Estimate the gravitational self-pressure at the surface by considering the gravitational force at the surface divided by the surface area, i.e. $P_g = F_g/A$.

b) Set this gravitational self-pressure equal the Fermi pressure and solve it for the mass. Express your result in terms of the proton mass m_p and the Planck mass $m_{\text{Planck}} = \sqrt{\frac{1}{G}}$, with G the gravitational constant.

   Hint: Since the system is electrically neutral, the number of electrons equals the number of protons. Assume that all mass M of the star is made up by protons.

c) Why does your result not depend on the electron mass?

d) Report your result in terms of the solar mass M_{sun}.

e) The critical mass for gravitational collapse is associated with the onset of relativistic degeneracy, $E_F = p_F \cdot c \geq m_e c^2$. Use the formula for the Fermi energy E_F given above to estimate the critical radius of a white dwarf.

Solution:

a) $P_g = F_g/A \sim \frac{G \cdot M^2}{R^2} \cdot \frac{1}{R^2} = \frac{G \cdot M^2}{R^4}$.

b) $\frac{G \cdot M^2}{R^4} = \left(\frac{N}{V}\right)^{4/3}, V = 4/3\pi \cdot R^3$. The number N of electrons equals the number of protons, since the system is electrically neutral. $N = \frac{M}{m_p}$.

$\frac{G \cdot M^2}{R^4} = \frac{1}{R^4} \frac{M}{m_p}^{4/3}$

$M^{2/3} = \frac{1}{G}/m_p^{4/3} \quad |(\ )^{3/2}$

$$M = \left(\frac{1}{G}\right)^{3/2} /m_p^2$$

$$M = m_{\text{Planck}}^3/m_p^2$$

N.B. The core usually consists of ^{12}C or ^{16}O nuclei and thus the mass of the white dwarf is twice the number of protons times the proton mass. This would lead to factor of 1/2 in the number of electrons, which is neglected here.

c) In the ultra-relativistic limit, $E_F = \sqrt{m_e^2 \cdot c^4 + p_e^2 \cdot c^2} \approx p_e \cdot c$.

d)

$$M = \frac{(2.2 \cdot 10^{-8})^3}{(1.67 \cdot 10^{-27})^2} \text{kg} = 3.82 \cdot 10^{-30} \text{kg} = 1.91 \cdot M_{\text{sun}}.$$

Precise result, including radiation pressure: $M = 1.39 \cdot M_{\text{sun}}$.

e) $\left(\frac{N}{V}\right)^{1/3} \geq m_e c^2$.

$\frac{1}{m_e c^2} \geq$, from above $N = \frac{M}{m_p} = m_{\text{Planck}}^3/m_p^3, V \sim R^3$.

$$R \leq \frac{m_{\text{Planck}}}{m_e c^2 m_p}$$

Inserting numbers results in

$$R \leq 200\text{MeV} \cdot \text{fm} \cdot 1.22 \cdot 10^{19}\text{GeV}/(0.5\text{MeV} \cdot 0.938\text{GeV}) = 200 \cdot 1.22/(0.5 \cdot 0.938) \cdot 10^4\text{m} = 5200\text{km}.$$

When the mass is higher than the critical mass, the star shrinks below the critical radius and collapses to form a neutron star or black hole.