Recitation 3 – Moderns Aspects of Nuclear Physics – SS22

Discussion on Fr., June 3, 2022, 11:15

3.1 Parity, C-, and G-parity in the $q\bar{q}'$ system (1 Point)

Consider the operators for parity inversion **P**, charge conjugation **C**, and G-parity $\mathbf{G} = \mathbf{C} \cdot e^{i\pi I_2}$ with eigenvalues $P, C, \eta_G = \pm 1$. Here, I_2 is the second component of the isospin vector $\vec{I}$.

a) Apply the charge conjugation operator **C** to the pion multiplet, i.e. π^+, π^0, π^-. Which states are eigenstates?

b) Apply the charge conjugation operator **G** to the pion multiplet, i.e. π^+, π^0, π^-. Which states are eigenstates?

c) Check that the relation for the eigenvalues $\eta_G = (-1)^{S+L+I}$ of the G-parity operator holds for the pion multiplett.

d) Consider states where the relationship $P = (-1)^J$ holds. Which spin S must these states posses?

e) Which values for the product CP of the above states are allowed?

f) Consequently, which states are forbidden? List three of them.

g) Look up the values $I^G(J^{PC})$ for the $\eta(0.549)$ meson, e.g. in the online version of the particle data book. Why is the decay into 3 pions not allowed in strong interactions? Why is the decay of the $\eta(0.549)$ meson into 2 pions not observed at all?

N.B. The flavor wave function of the $\eta(0.549)$ meson is given by $\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$, while for the π meson and ρ meson it is given by $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$.

Solution:

a) $\mathbf{C} = (-1)^{L+S}$, for the pions $L = S = 0$.

$$\mathbf{C}\begin{cases}\pi^+\\ \pi^0\\ \pi^-\end{cases} = \begin{cases}\pi^-\\ \pi^0\\ \pi^+\end{cases} \tag{1}$$

Only neutral particles can be eigenstates of the **C** operator, i.e. the neutral pion has eigenvalue $C = +1$.

b) $e^{i\pi I_2}$ is an additional rotation in isospin by multiples of π around the I_2 axis. The pions form an isospin triplet, i.e. $I = 1, I_3 = 1, 0, -1$.

$$\mathbf{G}\begin{cases}\pi^+\\ \pi^0\\ \pi^-\end{cases} = -\begin{cases}\pi^+\\ \pi^0\\ \pi^-\end{cases} \tag{2}$$

The minus sign enters through the rotation in isospin, e.g., the isospin component of the wave function can be described by the spherical harmonics $Y_{l=1}^{m=1,0,-1}$. A rotation of 180 degrees about the x-axis corresponds to $\theta \to \pi - \theta, \phi \to \pi - \phi$. Thus $Y_{l=1}^{m=1,0,-1} \to (-1)^I Y_{l=1}^{m=1,0,-1}$.

$\pi^0 : I^G(J^{PC}) = 1^-(0^{-+})$

$\pi^\pm : I^G(J^P) = 1^-(0^-)$

c) $\eta_G = (-1)^{S+L+I} = (-1)^{0+0+1} = -1$

d) From the lecture: $P = (-1)^{L+1}$.

Thus states with $P = (-1)^J$ must have spin $S = 1$. In case $S = 0$, it follows that $J = L$ and $P = (-1)^{J+1}$ and does violate the initial assumption.

e) $C = (-1)^{L+S}$, $P = (-1)^{L+1}$.

$CP = (-1)^{2L+2} = +1$

f) Consequently, states with $P = (-1)^J$ and $CP = -1$ are forbidden. These are $J^{PC} = 0^{+-}, 1^{-+}, 2^{+-}, 3^{-+}, ...$

g) $\eta(0.549)$: $I^G(J^{PC}) = 0^+(0^{-+})$

strong decay into three pions:

$\eta(0.549)$ has $G = +1$, the pions have $G = -1$. G is a multiplicative quantum number and thus needs an even number of pions in the final state, since G-parity is conserved in strong interactions.

decay into two pions:

$\eta(0.549)$ has $P = +1$, the pions have $P = -1$. In order to conserve parity, the pions must carry relative orbital momentum with an odd quantum number $l = 1, 3, 5, ...$ However, the $\eta(0.549)$ has total angular momentum $J = 0$, so do the pions. Thus, only $l = 0$ allowed. Therefore, no decay two two pions is allowed in strong and electromagnetic interactions.

3.2 Liquid drop formula (1 Point)

a) Consider the liquid drop formula. At a given mass number A, estimate the number of protons Z_{min} where the atomic mass $m(Z)$ reaches a minimum. Neglect the pairing energy. Draw your result in the $Z_{\text{min}}(N)$ plane, where $N = A - Z_{\text{min}}$. Also, draw a line $N = Z$.

b) Draw the nuclear binding energy $B(A, Z)/A$ as a function of the mass number A. Use the result $Z = Z_{\text{min}}$ from a). In what mass range does $B(A, Z)/A$ reach a maximum?

Solution:

Hint: In the literature, there are two slightly different terms for the asymmetry term, once using $(N-Z)^2$, and once using $(Z - A/2)^2$. This prefactor in the latter definition is a factor of 4 larger. Here, we use the former definition as given in the lecture with $a_{\text{sym}} = 23.3$ MeV.

The mass formula reads:

$M(A, Z) = Z \cdot m_H + (A - Z) \cdot m_n - B(A, Z)$, with the binding energy from the liquid drop model

$B(A, Z) = a_V A - a_O A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_{\text{sym}} \frac{(N-Z)^2}{A} + \frac{\delta}{A^{1/2}}$.

Since $A = $ const., express N trough A, Z.

$B(A, Z) = a_V A - a_O A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_{\text{sym}} \frac{(A-2Z)^2}{A} + \frac{\delta}{A^{1/2}}$.

a) We take the partial derivative of the mass formula with respect to Z and determine the proton number $Z_{\rm min}$ where the derivative vanishes.

$$\frac{\partial m(Z,A)}{\partial Z} = m_H - m_n + \cdot a_{\rm sym} \cdot \frac{-4(A-2Z)}{A} + 2a_C \frac{Z}{A^{1/3}} \tag{3}$$
$$= m_H - m_n + a_{\rm sym} \cdot \frac{-4A+8Z}{A} + 2a_C \frac{Z}{A^{1/3}} \tag{4}$$
$$= m_H - m_n - 4a_{\rm sym} + \left(\frac{8a_{\rm sym}}{A} + \frac{2a_C}{A^{1/3}}\right) Z \tag{5}$$
$$= m_H - m_n - 4a_{\rm sym} + 2\left(\frac{4a_{\rm sym}}{A} + \frac{a_C}{A^{1/3}}\right) Z = 0 \tag{6}$$
$$Z_{\rm min} = \frac{A}{2} \frac{m_n - m_H + 4a_{\rm sym}}{4a_{\rm sym} + a_C A^{2/3}} \tag{7}$$
$$\tag{8}$$

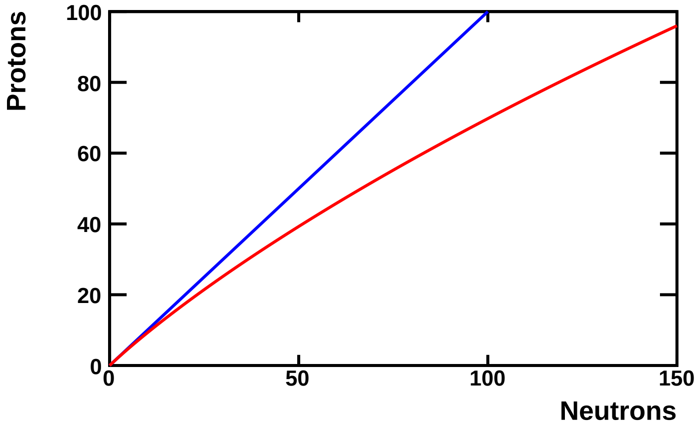
with $m_n - m_H = 0.7$ MeV, $a_C = 0.71$ MeV und $a_{\rm sym} = 23.3$ MeV, it follows

$$= \frac{A}{1.9725 + 0.01503 A^{2/3}} \tag{9}$$

Calculate for a fixed value of A with $A = 1 - 300$ the value for $Z_{\rm min}$ with $N = A - Z_{\rm min}$. One gets the following numbers:

A	$Z_{\rm min}(N)$	N	element
16	7.7	8.3	$^{16}_{8}$O
62	28.1	33.9	$^{62}_{28}$Ni
100	43.5	56.5	$^{100}_{44}$Ru
208	83.2	124.8	$^{208}_{82}$Pb

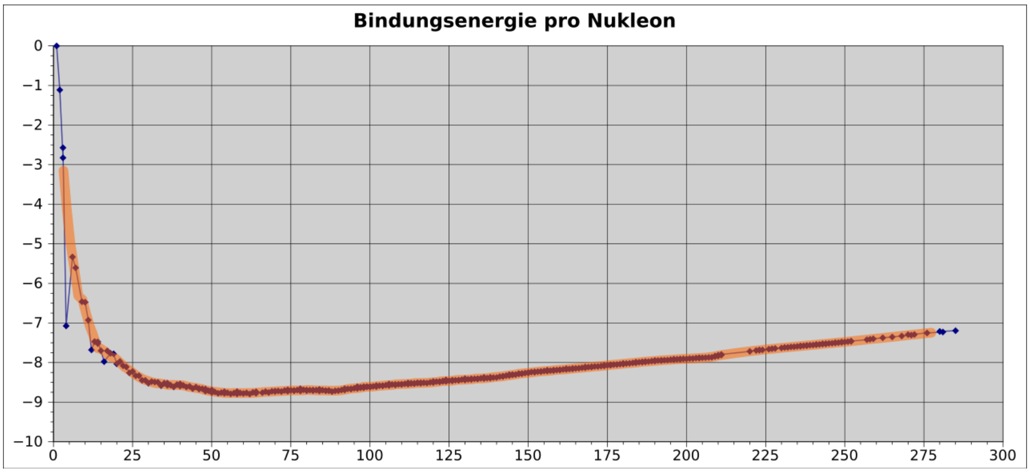
The line of stability (red) sits below the bisecting line (blue), the difference increases with increasing N.



b) Here, the nuclear binding energy is divided by A:

$$\frac{B(Z,A)}{A} = a_V - a_O A^{-1/3} - a_C \frac{Z^2}{A^{4/3}} - a_{\rm sym} \frac{(A-2Z)^2}{A^2} \tag{10}$$

Now, the result for $Z_{\rm min}$ from a) is inserted. This results in the plotted curve. The most stable nuclei are located in the mass region $A \approx 60$, as experimentally observed. The most stable nucleus is ^{62}Ni (http://dx.doi.org/10.1119/1.17828). Hint: The plot shows the $-\frac{B(Z,A)}{A}$ plane.



3.3 Shell model and spin orbit coupling (1 Point)

Spin orbit coupling is considered by an additional term $V_{ls} = \text{const.} \cdot (\hat{\vec{l}} \cdot \hat{\vec{s}})$ in the nuclear potential $V = V(r)$. This term leads to a splitting of the otherwise degenerated states with $j = l \pm 1/2$.

a) Show that the contribution ΔE_{jls} to the binding energy due to spin-orbit coupling is

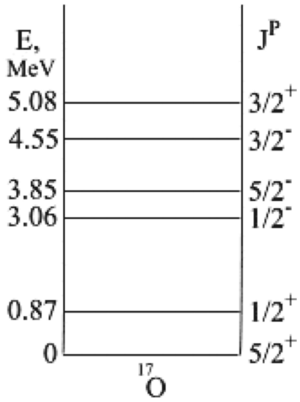
$$\Delta E_{jls} = \frac{a}{2}[j(j+1) - l(l+1) - s(s+1)]$$

with a the spin-orbit coupling constant.

b) Write down the nucleon configuration of protons and neutrons in the nucleus ^{17}O in the shell model. What values for spin and parity (J^π) do you expect for the ground state of ^{17}O in the shell model?

c) Which excited states given in the Figure can be described within the shell model?

d) Which of the excited states shown is partner to the ground state with identical orbital momentum l? Determine the spin-orbit coupling constant a. Use the energies (E) given in the Figure.



Solution:

a) To calculate ΔE_{LS} one needs to find the matrix element:

$$\Delta E_{jls} = \langle jlsm_j | (\hat{\vec{l}} \cdot \hat{\vec{s}}) | jlsm_j \rangle$$

Since

$$\hat{\vec{j}}^2 = (\hat{\vec{l}} + \hat{\vec{s}})^2 = \hat{\vec{l}}^2 + 2(\hat{\vec{l}} \cdot \hat{\vec{s}}) + \hat{\vec{s}}^2$$

we can write

$$(\hat{\vec{l}} \cdot \hat{\vec{s}}) = \frac{1}{2}(\hat{\vec{j}}^2 - \hat{\vec{l}}^2 - \hat{\vec{s}}^2)$$

Knowing the eigenvalues of the operators:

$$\hat{\vec{j}}^2 \left|jls\right\rangle = j(j+1)\left|jls\right\rangle ; \quad \vec{l}^2 \left|jls\right\rangle = l(l+1)\left|jls\right\rangle ; \quad \vec{s}^2 \left|jls\right\rangle = s(s+1)\left|jls\right\rangle$$

we obtain

$$\Delta E_{jls} = \langle jlsm_j| \, (\hat{\vec{l}} \cdot \hat{\vec{s}}) \, |jlsm_j\rangle = \frac{a}{2}[j(j+1) - l(l+1) - s(s+1)]$$

b) The ground state of ^{17}O corresponds to a single neutron above the closed N=8, Z=8 shell

$$\left|1s_{1/2}\right\rangle^4 \left|1p_{3/2}\right\rangle^8 \left|1p_{1/2}\right\rangle^4 \left|1d_{5/2}\right\rangle_n^1$$

Closed shells do not contribute to the orbital momentum, they have $l = 0$, and parity $P = +1$. Hence $j = 5/2$ and $\pi = (-1)^{(l=2)}$: $J^\pi = 5/2^+$.

c) The $|1d_{5/2}\rangle$ neutron can be excited to next sub-shells $2s_{1/2}$ and $1d_{3/2}$, which correspond to $1/2^+$ and $3/2^+$ states, respectively.

Discussion: the nature of other states is more complicated. For instance the level $1/2^-$ at 3.06 MeV corresponds to an excitation of a nucleon from $1p_{1/2}$ to $1d_{5/2}$ coupling to the total angular moment $j = 0$. The spin-parity of the state in this case is determined by the hole in $1p_{1/2}$, that is $J^\pi = 1/2^-$.

d) The spin-orbit interaction causes splitting with $j = l \pm 1/2$. In the figure, these are the states $1d_{5/2}$ with $E = 0$ and $1d_{3/2}$ at $E = 5.08$ MeV.

$$\Delta E(1d_{5/2}) - \Delta E(1d_{3/2}) = \frac{a}{2}\left[\frac{5}{2} \cdot \frac{7}{2} - \frac{3}{2} \cdot \frac{5}{2}\right] = \frac{5a}{2} = -5.08 \text{ MeV}$$

Hence $a \approx -2$ MeV.
