Recitation 1 – Moderns Aspects of Nuclear Physics – SS22

Discussion on Fr., May 6, 2022, 11:15

1.1 Magnetic moment of the proton and neutron (1 Point)

A quark that behaves like a Dirac particle has a magnetic dipole moment $\langle\vec{\mu}\rangle$. The magnetic dipole operator $\vec{\mu}$ is defined by

$$\vec{\mu} = g_s\mu_0\vec{s} = g_s\mu_0\frac{\vec{\sigma}}{2}, \quad (\vec{s} = \frac{1}{2}\vec{\sigma})$$

with g_s the g factor, μ_0 the quark magneton, $\vec{s}$ the spin of the quarks, and $\vec{\sigma}$ the vector of the Pauli spin matrices $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

The magnetic dipole moment $\langle\vec{\mu}\rangle$ of the quark is defined as the expectation value of the magnetic dipole operator in the spin-up state, i.e. $\langle\vec{\mu}\rangle = \langle\uparrow|\vec{\mu}|\uparrow\rangle = \langle\uparrow|\mu_z|\uparrow\rangle = \frac{g_s\mu_0}{2}$. The dipole moment $\langle\vec{\mu}_u\rangle$ of the up quark is thus
$\langle\vec{\mu}_u\rangle = \frac{g_s\mu_N}{2}\frac{m_N}{m_u}\frac{2}{3}$, and the dipole moment $\langle\vec{\mu}_d\rangle$ of the down quark $\langle\vec{\mu}_d\rangle = \frac{g_s\mu_N}{2}\frac{m_N}{m_d}(-\frac{1}{3})$, with m_N the mass of the proton and m_u, m_d the masses of the up and down constituent quarks.

The magnetic dipole moment $\langle\vec{\mu}_p\rangle$ of the proton is defined as the expectation value of the sum of the magnetic dipole operator of the three valence quarks in the proton spin-up state,

$$\langle\vec{\mu}_p\rangle = \langle p\uparrow|\sum_{j=1}^{3}\vec{\mu}_j\,|p\uparrow\rangle = \langle p\uparrow|\sum_{j=1}^{3}\mu_z|p\uparrow\rangle$$

with $\vec{\mu}_j$ the dipole operator acting on the valence quark j.

In order to calculate the magnetic dipole moment $\langle\vec{\mu}_p\rangle$ of the proton, one needs the spin-flavor wave function that can be written as

$$\begin{aligned}|p\uparrow\rangle = \frac{1}{\sqrt{18}}(2|u\uparrow d\downarrow u\uparrow\rangle &+ 2|u\uparrow u\uparrow d\downarrow\rangle + 2|d\downarrow u\uparrow u\uparrow\rangle \\ -|u\uparrow u\downarrow d\uparrow\rangle &- |u\downarrow d\uparrow u\uparrow\rangle - |u\uparrow d\uparrow u\downarrow\rangle \\ -|d\uparrow u\downarrow u\uparrow\rangle &- |d\uparrow u\uparrow u\downarrow\rangle - |u\downarrow u\uparrow d\uparrow\rangle\end{aligned} \tag{1}$$

a) Calculate the magnetic dipole moment $\langle\vec{\mu}_p\rangle$ of the proton.
Hint:: The individual terms in the wave functions are orthogonal to each other. Use valence quark masses.

b) Now, calculate the magnetic dipole moment $\langle\vec{\mu}_n\rangle$ of the neutron. First, write down the wave function of the neutron unser the assumption of isospin symmetry, i.e. exchange the up quarks for down quarks and vice versa.

c) Calculate the ratio of the magnetic moments of the proton and neutron. Compare your results to experimental results.

d) Is the given spin-flavour wave function symmetric or antisymmetric under the exchange of identical particles? Argue why.

Solution:

a) The magnetic dipole moment of the proton is

$$\langle\vec{\mu}_p\rangle = \langle p\uparrow|\sum_{j=1}^{3}\mu_z|p\uparrow\rangle$$

The masse of the up and down quark are assumed to be equal.

The first term of the wave function (with $m_u = m_d$) results in

$$\langle|u\uparrow d\downarrow u\uparrow|\sum_{j=1}^{3}\mu_z|u\uparrow d\downarrow u\uparrow\rangle = \langle\mu_u\rangle - \langle\mu_d\rangle + \langle\mu_u\rangle = \frac{g_s\mu_N}{2}\frac{m_N}{m_u}\frac{5}{3}. \quad (2)$$

This leads to

$$\langle\vec{\mu}_p\rangle = \frac{g_s\mu_N}{2}\frac{m_N}{m_u}\frac{1}{18}(4\frac{5}{3} + 4\frac{5}{3} + 4\frac{5}{3} - 6\frac{1}{3}) = \frac{g_s\mu_N}{2}\frac{m_N}{m_u} \quad (3)$$

and

b)

$$\langle\vec{\mu}_n\rangle = \frac{g_s\mu_N}{2}\frac{m_N}{m_u}(-\frac{2}{3})$$

and finally

c)

$$\frac{\langle\vec{\mu}_p\rangle}{\langle\vec{\mu}_n\rangle} = -\frac{3}{2},$$

in good agreement with the experimental value of -1.46.

d) The spin-flavour wave function is symmetric, the quarks are in the ground state with vanishing orbital momentum, $l = 0$. Thus, the spatial part of the wave function is symmetric. This necessitates the introduction of an additional degree of freedom, i.e. color, with the color wave function to be antisymmetric.

N.B. The spin wave function $\chi_{0,0}(1,2)$ for the singlet state is antisymmetric, the spin wave function $\chi_{1,m_z=1,0,-1}(1,2)$ for the triplet state is symmetric.

$$\chi_{0,0}(1,2) = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$$

$$\chi_{1,1}(1,2) = |\uparrow\uparrow\rangle$$
$$\chi_{1,0}(1,2) = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle]$$
$$\chi_{1,-1}(1,2) = |\downarrow\downarrow\rangle$$

1.2 Proton form factor (1 Point)

The experimental result shown in figure 1 shows that the electrical form factor of the proton is well parametrised by a 'dipole function':

$$G_E(Q^2) = \frac{1}{\left(1 + Q^2/Q_0^2\right)^2}, \quad \text{with } Q_0^2 = 0.71\ \text{GeV}^2.$$

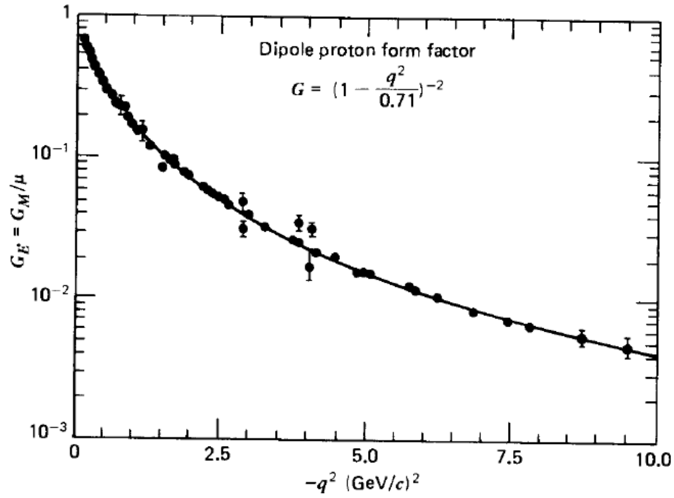


Figure 1: Experimental data on the electrical form factor of the proton.

a) Taking $Q^2 \approx -\vec{q}^{\,2}$, show that this implies that the proton has an exponential charge distribution of the form

$$\rho(\vec{r}) = \rho_0 e^{-r/a} \,.$$

Find the value for a.

b) For a spherically symmetric charge distribution $\rho(r)$, where

$$\int \rho(r) d^3\vec{r} = 1 \,,$$

show that the form factor can be expressed as

$$\begin{aligned} F(\vec{q}^{\,2}) &= \frac{4\pi}{q} \int_0^\infty r \sin(qr) \rho(r) \, dr \\ &\approx 1 - \frac{1}{6} q^2 \langle R^2 \rangle + \dots \,, \end{aligned}$$

where $\langle R^2 \rangle$ is the mean square charge radius.
Hint: You will need to use the expansion $\sin(qr) \approx qr - \frac{1}{3!}(qr)^3 + \dots$.
Hence show that

$$\langle R^2 \rangle = -6 \left[\frac{dF(\vec{q}^{\,2})}{dq^2} \right]_{q^2=0} \,.$$

Estimate $\langle R^2 \rangle$ of the proton.

c) Plot the charge distribution of a proton and indicate the values of a and $\sqrt{\langle R^2 \rangle}$. Compare these values with the proton radius given by the Particle Data Group.

d) What is the optimal angle for measuring scattered electrons in order to obtain a high sensitivity to the proton radius?

Solution:
We start with the general consideration on the form factor of a spherically symmetric charge distribution.

$$F(\vec{q}^{\,2}) = \int \rho(\vec{r}) e^{i\vec{q}\cdot\vec{r}} d^3\vec{r}$$

Using spherical coordinates leads to

$$F(\vec{q}^2) = 2\pi \int_0^\infty dr\, r^2 \int_{-1}^1 d(\cos\theta)\rho(\vec{r})e^{i|\vec{q}||\vec{r}|\cos\theta}$$

$$F(\vec{q}^2) = 2\pi \int_0^\infty dr\, r^2 \rho(\vec{r}) \frac{e^{i|\vec{q}||\vec{r}|} - e^{-i|\vec{q}||\vec{r}|}}{i|\vec{q}||\vec{r}|}$$

$$F(\vec{q}^2) = \frac{4\pi}{|\vec{q}|} \int_0^\infty dr\, r\rho(\vec{r}) \sin(|\vec{q}||\vec{r}|)$$

a) Using $\rho(\vec{r}) = \rho_0 e^{-r/a}$, one gets (e.g. from Wolframalpha)

$$F(\vec{q}^2) = \frac{4\pi\rho_0}{|\vec{q}|} \frac{2a^3|\vec{q}|}{(1+a^2|\vec{q}^2|)^2}$$

$$F(\vec{q}^2) = 8\pi a^3 \rho_0 \frac{1}{(1+\frac{|\vec{q}|}{Q_0}^2)^2},$$

with $a^2 = \frac{1}{Q_0^2}$.
Thus,

$$a = \frac{1}{\sqrt{0.71\text{GeV}^2}} = \frac{1}{0.84\text{GeV}}$$

$$a = \frac{0.2}{0.84}\ \text{fm} = 0.24\ \text{fm}.$$

b) We start with

$$F(\vec{q}^2) = \frac{4\pi}{q} \int_0^\infty r \sin(qr)\rho(r)\, dr$$

Taylor expansion of the sine term leads to

$$= 4\pi \int_0^\infty dr\, r^2 \rho(r) \left[1 - \frac{q^2 r^2}{3!} + \frac{q^4 r^4}{5!} + ...\right].$$

Taking the derivative with respect to dq^2, at $q^2 = 0$ only the term $-\frac{r^2}{3!}$ remains, so that

$$\frac{dF(\vec{q}^2)}{dq^2}|_{q^2=0} = -\frac{4\pi}{6}\int_0^\infty dr\, r^4 \rho(r) = -\frac{e}{6}\langle r^2\rangle.$$

Taking the dipole form factor,

$$\frac{dF(\vec{q}^2)}{dq^2} = -\frac{2Q_0^4}{(Q_0^2+\vec{q}^2)^3}$$

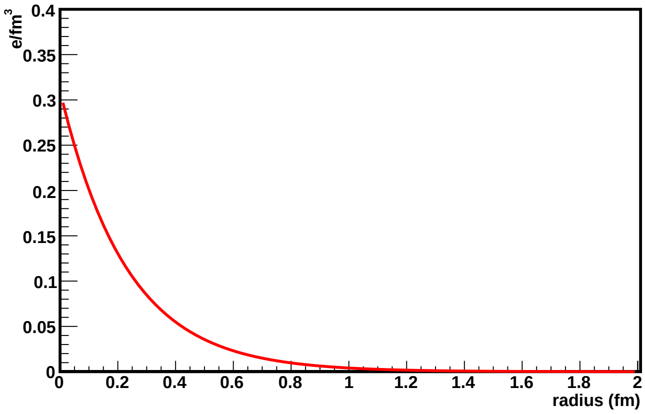
$$\frac{dF(\vec{q}^2)}{dq^2}|_{q^2=0} = -\frac{2}{Q_0^2}$$

Evaluate

$$\langle R^2\rangle = -6\left[\frac{dF(\vec{q}^2)}{dq^2}\right]_{q^2=0} = \frac{12}{Q_0^2} = 12a^2$$

$$\sqrt{\langle R^2\rangle} = \sqrt{12}a = 0.83\ \text{fm}.$$

c) This value is in agreement with the PDG value



d) One has to measure small values of $\vec{q}^{\,2}$, i.e. small momentum transfers and thus at small scattering angles.

1.3 Quark-Parton Model (1 Point)

In this problem we discuss structure functions in deep-inelastic scattering in the quark-parton model. Figure 2 shows measurements of the structure functions in electron-proton (F_2^{ep}) and electron-deuteron (F_2^{eD}) scattering made at the Stanford Linear Accelerator Center (SLAC). From *ep* and *eD* scattering one can determine the Parton Distribution Functions for proton (F_2^{ep}) and neutron (F_2^{en}). Consider only u, $\bar{u}$, d, and $\bar{d}$ quarks for this problem.

a) Write down the structure functions F_2^{ep} and F_2^{en} in the quark-parton model in terms of the parton distributions $q_i^{\mathrm{p}}(x)$ of the proton.
Hint: the master formula of the quark-parton model is

$$F_2^{\mathrm{ep}} = x \sum_i Q_i^2 q_i^{\mathrm{p}}$$

b) Determine the quark-parton model prediction for

$$R = \frac{\int_0^1 F_2^{\mathrm{eD}}(x)\,\mathrm{d}x}{\int_0^1 F_2^{\mathrm{ep}}(x)\,\mathrm{d}x}.$$

Experimentally one finds $R \approx 0.84$. What does this imply for the ratio $f_{\mathrm{d}}/f_{\mathrm{u}}$ where $f_{\mathrm{q}} := \int_0^1 x(q(x) + \bar{q}(x))\,\mathrm{d}x$?

c) Assume $f_{\mathrm{u}} = 0.36$. What fraction of the proton's momentum is carried by quarks?

d) Show that

$$\int\limits_0^1 \frac{\left[F_2^{\mathrm{ep}}(x) - F_2^{\mathrm{en}}(x)\right]}{x}\,\mathrm{d}x \approx \frac{1}{3} + \frac{2}{3}\int\limits_0^1 \left[\bar{u}(x) - \bar{d}(x)\right]\,\mathrm{d}x.$$

Interpret the measured value of 0.24 ± 0.03.

e) In the limit $x \to 1$ valence quarks are expected to dominate. Write the ratio

$$\frac{F_2^{\mathrm{en}}(x)}{F_2^{\mathrm{ep}}(x)}$$

in this limit in terms of the valence quark distributions $u_{\mathrm{v}}(x)$ and $d_{\mathrm{v}}(x)$. Experimentally one finds $F_2^{\mathrm{en}}(x)/F_2^{\mathrm{ep}}(x) \to 0.25$ for $x \to 1$. What does this imply for the ratio $d_{\mathrm{v}}(x)/u_{\mathrm{v}}(x)$?

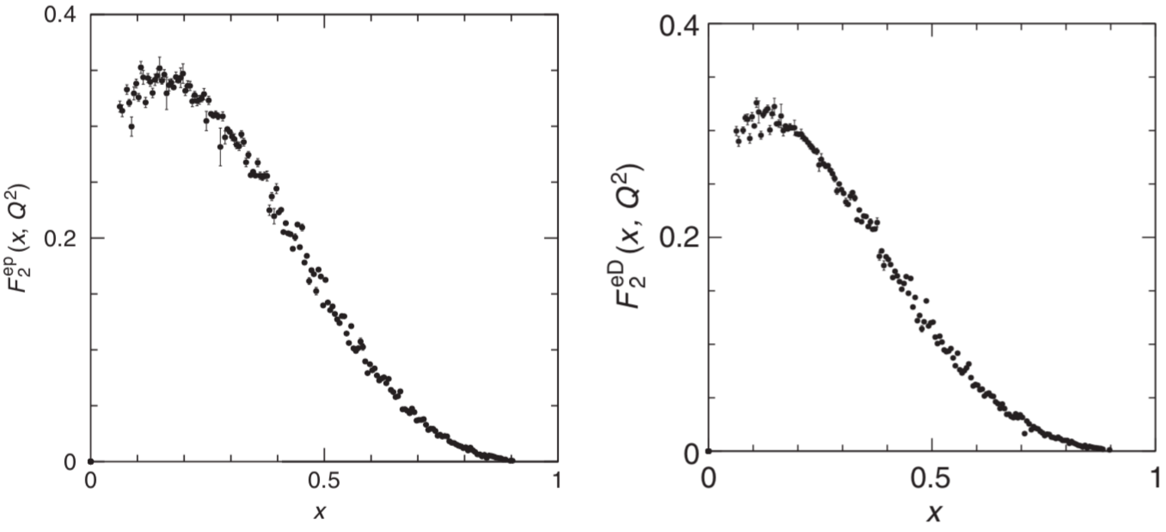


Figure 2: $F_2^{\mathrm{ep}}(x)$ (a) and $F_2^{\mathrm{eD}}(x)$ (b) measured for $2 < Q^2/\mathrm{GeV}^2 < 30$. Data from Whitlow et al., Phys. Lett. B282 (1992) 475.

Solution:

a) The master formula of the quark-parton model is

$$F_2^{\mathrm{ep}} = x \sum_i Q_i^2 q_i^{\mathrm{p}}$$

This yields

$$F_2^{\mathrm{ep}}(x) = x \left(\frac{4}{9} \left[u(x) + \bar{u}(x) \right] + \frac{1}{9} \left[d(x) + \bar{d}(x) \right] \right)$$

$$F_2^{\mathrm{en}}(x) = x \left(\frac{4}{9} \left[d(x) + \bar{d}(x) \right] + \frac{1}{9} \left[u(x) + \bar{u}(x) \right] \right)$$

b) We have

$$F_2^{\mathrm{eD}}(x) = \frac{F_2^{\mathrm{ep}}(x) + F_2^{\mathrm{en}}(x)}{2} = x \frac{5}{18} \left(u(x) + \bar{u}(x) + d(x) + \bar{d}(x) \right)$$

Hence we obtain

$$R = \frac{\int_0^1 F_2^{\mathrm{eD}}(x)\,\mathrm{d}x}{\int_0^1 F_2^{\mathrm{ep}}(x)\,\mathrm{d}x} = \frac{\frac{5}{18}(f_u + f_d)}{\frac{4}{9} f_u + \frac{1}{9} f_d} = \frac{5(f_u + f_d)}{8 f_u + 2 f_d} \equiv \frac{5(1+r)}{8+2r}, \quad r := f_d/f_u.$$

so that

$$r = \frac{5 - 8R}{2R - 5} \approx 0.52$$

c) With the above results we get

$$f_{\mathrm{quarks}} = f_u + f_d \approx 1.5 f_u = 0.55$$

The quarks carry only about 50% of the proton's momentum, the rest is carried by gluons.

d)

$$\int_0^1 \frac{[F_2^{\text{ep}}(x) - F_2^{\text{en}}(x)]}{x}\,\mathrm{d}x = \frac{1}{3}\int_0^1 u(x) + \bar{u}(x) - (d(x) + \bar{d}(x))\,\mathrm{d}x$$

$$= \frac{1}{3}\int_0^1 u(x) - \bar{u}(x) + 2\bar{u}(x) - (d(x) - \bar{d}(x) + 2\bar{d}(x))\,\mathrm{d}x$$

$$\approx \frac{1}{3} + \frac{2}{3}\int_0^1 \bar{u}(x) - \bar{d}(x)\,\mathrm{d}x$$

The measured value of 0.24 ± 0.03 is smaller than $1/3$, indicating that there are more $\bar{d}$ quarks than $\bar{u}$ quarks in the proton.

e) Ignoring the sea quarks for $x \to 1$ we obtain

$$F_2^{\text{ep}}(x) = x\frac{1}{9}\left(4u_V(x) + d_V(x)\right), \quad F_2^{\text{en}}(x) = x\frac{1}{9}\left(u_V(x) + 4d_V(x)\right)$$

This gives

$$R = \frac{F_2^{\text{en}}(x)}{F_2^{\text{ep}}(x)} = \frac{u_V(x) + 4d_V(x)}{4u_V(x) + d_V(x)} = \frac{1 + 4d_V(x)/u_V(x)}{4 + d_V(x)/u_V(x)}$$

The measured value for the ratio of 0.25 seems to imply that $d_V(x)/u_V(x)$ vanishes as x approaches 1.