5. Neutrino oscillations

If neutrinos have masses and lepton flavors are mixed in the weak CC interactions, lepton flavor is not conserved in neutrino propagation.

This phenomenon is usually referred to as neutrino oscillations. In brief, a weak eigenstates, ν_α, which by default is the state produced in the weak CC interaction of a charged lepton l_α , is the linear combination determined by the mixing matrix U

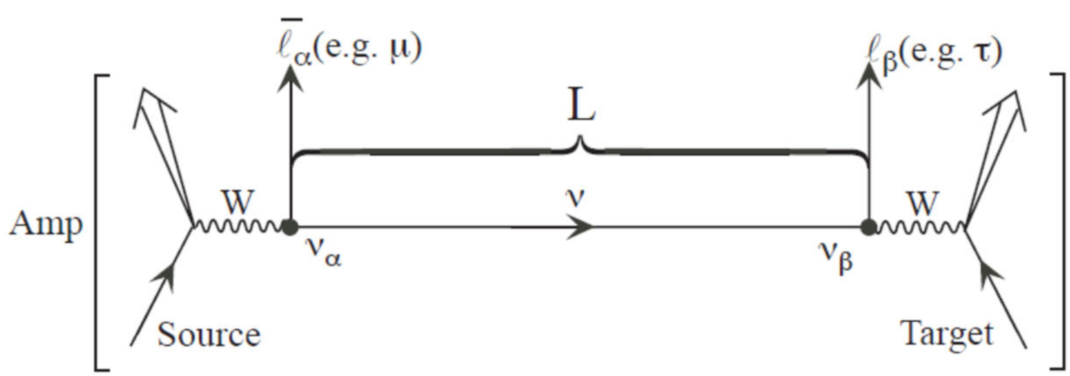
$$|\nu_\alpha\rangle = \sum_{i=1}^{n} U^*_{\alpha i}|\nu_i\rangle \,,$$

ν_i are the mass eigenstates

5.1 Neutrino oscillation in vacuum

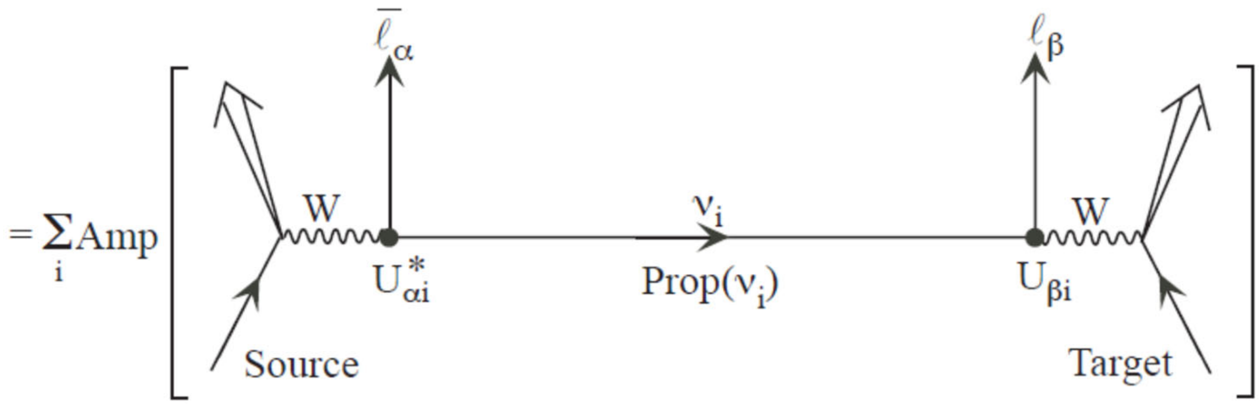
(follows a derivation by Boris Kayser)

A neutrino produced with flavor ν_α in the source can thus interact as ν_β in the target:



B.Kayser

Neutrino propagation: mass states



B.Kayser

Assumption:

Coherent mass states propagate as plane waves: $|\nu_i(t,x)\rangle = |\nu_i(0)\rangle \exp\left(-ip_\mu x^\mu\right)$

The amplitude $\mathcal{A}\left(\nu_\alpha \to \nu_\beta\right)$ for oscillation, i.e. the amplitude that a ν_α produced in the source is detected as ν_β in the target is given by:

$$\mathcal{A}\left(\nu_\alpha \to \nu_\beta\right) \sim \sum_i \mathcal{A}\left(W \to \bar{\ell}_\alpha \nu_i\right) \text{Propagator}\left(\nu_i\right) \mathcal{A}\left(\nu_i \to \ell_\beta W\right)$$

$$\sim \sum_i \mathrm{U}^*_{\alpha i} \cdot \text{Propagator}\left(\nu_i\right) \cdot \mathrm{U}_{\beta i}$$

The propagator $\sim \exp\left(ip_\mu x^\mu\right)$ describes the neutrino propagation along the distance L and is given in the lab frame by $\exp\left(-i(E_i t - p_i L)\right)$, where t is the flight time from the source to the target at distance L. E_i and p_i are the energy and the momentum in the lab frame. That is, each mass eigenstate ν_i picks up the phase factor $\phi_i = -i(E_i t - p_i L)$

As the oscillation probability is given by $\mathcal{P}_{\alpha\beta} = \left|\mathcal{A}\left(\nu_\alpha \to \nu_b\right)\right|^2$ only the relative phase differences between the different propagation phases are relevant:

$$\Delta\phi_{ij} = -(E_i t - p_i L) + (E_j t - p_j L) = \left(p_i - p_j\right)L - \left(E_i - E_j\right)t$$

In practice experiments do not measure t. Instead t is replaced by $L / \bar{v}$ where $\bar{v}$ is the average velocity of the 2 neutrinos mass states.

$$\bar{v} = \frac{p_1 + p_2}{E_1 + E_2}$$

One obtains for the phase difference:

$$\Delta\phi_{ij} = \frac{p_i^2 - p_j^2}{p_i + p_j}L - \frac{E_i^2 - E_j^2}{p_i + p_j}L = \frac{m_j^2 - m_i^2}{p_i + p_j}L = \frac{\Delta m_{ij}^2}{2E}L$$

where we have used that for highly relativistic neutrinos p_1 and p_2 can be approximated by the neutrino beam energy $E_1 \approx E_1 \approx E$ (minor differences play no role in the sum).

Thus the relative phases in $\mathcal{A}\left(\nu_\alpha \to \nu_\beta\right)$ between the neutrinos are correct if we take as propagator:

$$\text{Propgator}\left(\nu_i\right) = \exp\left(i m_i^2 \frac{L}{2E}\right)$$

For the transition amplitude $\mathcal{A}\left(\nu_\alpha \to \nu_\beta\right)$ one thus obtains:

$$\mathcal{A}\left(\nu_\alpha \to \nu_\beta\right) = \sum_i U_{\alpha i}^* \exp\left(i m_i^2 \frac{L}{2E}\right) U_{\beta i}$$

The oscillation probability $\mathcal{P}\left(\nu_\alpha \to \nu_\beta\right)$ is obtained from $\left|\mathcal{A}\left(\nu_\alpha \to \nu_\beta\right)\right|^2$ and exploiting unitarity:

$$\mathcal{P}\left(\nu_\alpha \to \nu_\beta\right) = \left|\mathcal{A}\left(\nu_\alpha \to \nu_\beta\right)\right|^2 = \delta_{\alpha\beta} - 4\sum_{i,j:i>j} \Re\left\{U_{\alpha i}^* U_{\beta i} U_{\alpha j}^* U_{\beta j}\right\} \sin\left(\Delta m_{ij}^2 \frac{L}{2E}\right)$$

$$+\, 2\sum_{i,j:i>j} \Im\left\{U_{\alpha i}^* U_{\beta i} U_{\alpha j}^* U_{\beta j}\right\} \sin\left(\Delta m_{ij}^2 \frac{L}{2E}\right)$$

For anti-neutrinos:

where $\Delta m_{ij}^2 = m_j^2 - m_i^2$

$\mathcal{P}\left(\bar{\nu}_\alpha \to \bar{\nu}_\beta\right) = \ldots$ with "– "-sign (complex conj. PMNS matrix elements)

Excursus: Majorana-Phases:

For Majorana-neutrinos:

$$U_{PMNS} = V_{PMNS} \cdot \mathrm{diag}\left(e^{i\eta_1}, e^{i\eta_2}, 1\right) = \begin{pmatrix} e^{i\eta_1}U_{e1} & e^{i\eta_2}U_{e2} & U_{e3} \\ e^{i\eta_1}U_{\mu 1} & e^{i\eta_2}U_{\mu 2} & U_{\mu 3} \\ e^{i\eta_1}U_{\tau 1} & e^{i\eta_2}U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

the additional phases leave the combinations $U^*_{\alpha i}U_{\beta i}U^*_{\alpha j}U_{\beta j}$ invariant.

The Majorana-phases do not change mixing (no additional CP violation)

Majorana phases are only observable in processes which change the lepton number by two units. Neutrino mixing changes the flavor $\rightarrow$ Majorana phases not visible.

CP-violation in neutrino mixing:

$$J_{CP} = \Im\left\{U^*_{\alpha i} U_{\beta i} U^*_{\alpha j} U_{\beta j}\right\} \neq 0 \quad \Longleftrightarrow \quad \mathcal{P}\left(\nu_\alpha \to \nu_\beta\right) \neq \mathcal{P}\left(\bar{\nu}_\alpha \to \bar{\nu}_\beta\right)$$

i.e. if U_{PMNS} is complex ($\delta_{CP} \neq 0, \pi$).

Usage of SI-units:

The expression $\left(\Delta m^2_{ij} \frac{L}{2E}\right)$ uses natural units. Use $\hbar$ and c to transform to SI-units:

$$\Delta m^2_{ij} \frac{L}{2E} \to 1.27 \cdot \Delta m^2_{ij}\left[\mathrm{eV}^2\right] \frac{L[\mathrm{km}]}{2E[\mathrm{GeV}]}$$

Example: Experiments studying 1 GeV neutrinos travelling $L \approx 10^4$ km is sensitive to m_{ij}^2 – splitting as small as ~10^{-4} eV^2 (Sensitivity of atmospheric neutrinos passing the earth)

Some remarks on the derivation of the mixing formula:

Many text books use either equal energy or equal momentum assumption:

$$\Delta\phi_{ij} = -(E_i t - p_i L) + (E_j t - p_j L) = \left(p_i - p_j\right)L - \left(E_i - E_j\right)t$$

Equal energy: $E_i = E_j = E$ and $p_i = \sqrt{E^2 - m_i^2} = E - \frac{m_i^2}{2E}$

$$\Delta\phi_{ij} = \left(p_i - p_j\right)L - \left(E_i - E_j\right)t = \frac{m_i^2 - m_j^2}{2E}L = \frac{\Delta m_{ij}^2}{2E}L$$

Equal momentum: $p_i = p_j = p$ and $E_i = \sqrt{p^2 + m_i^2} = p - \frac{m_i^2}{2p}$

$$\Delta\phi_{ij} = \left(p_i - p_j\right)L - \left(E_i - E_j\right)t = \frac{-\left(m_i^2 - m_j^2\right)}{2p}t = \frac{\Delta m_{ij}^2}{2E}L$$

(where in the last equality L=ct and pc=E has been used)

It turns out that neither the equal momentum nor the equal energy ansatz is correct (see e.g. E. Akhmedov arXiv:1901.05232v1)

Most derivations (including ours) use a plane-wave treatment for the propagation of the neutrino - instead a wave-package ansatz is needed (see arXiv:1901.05232v1) However, a correct treatment using wave-packages results in the same formula.

Special case: Two-neutrino mixing

For two neutrino generations the PMNS-mixing matrix is a unitary 2×2 matrix:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

(1 parameter=mixing angle)

There is only one mass difference Δm^2 and the flavor mixing probabilities are the same:

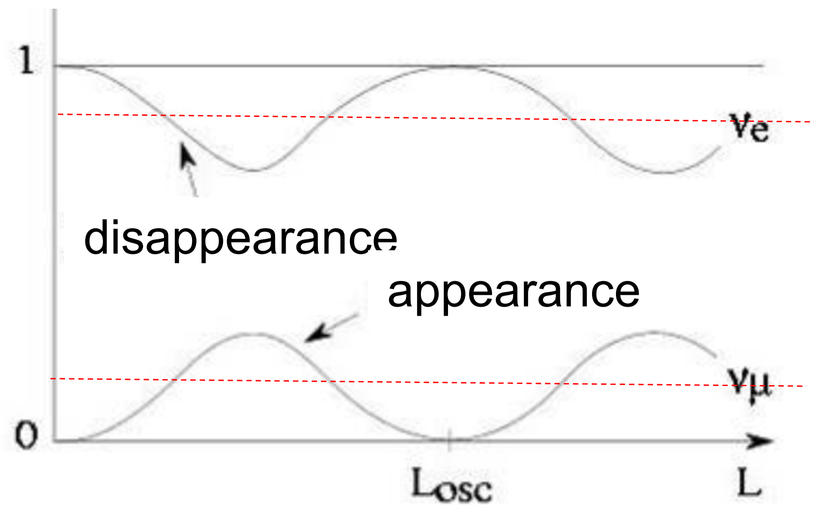
$$\mathcal{P}(\nu_e \to \nu_\mu) = \mathcal{P}(\nu_\mu \to \nu_e) = \mathcal{P}(\bar{\nu}_e \to \bar{\nu}_\mu) = \mathcal{P}(\bar{\nu}_\mu \to \bar{\nu}_e)$$

with
$$\mathcal{P}(\nu_e \to \nu_\mu) = \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2}{4E}L\right) = 1 - \mathcal{P}(\nu_e \to \nu_e)$$

Introducing the oscillation length L_0

$$L_{osc}[\mathrm{m}] = 4\pi\hbar c\frac{E}{\Delta m^2} = 2.48 \cdot \frac{E[\mathrm{GeV}]}{\Delta m^2[\mathrm{eV}^2]}$$

One can rewrite
$$\sin^2\left(\frac{\Delta m^2}{4E}L\right) = \sin^2\left(\pi\frac{L}{L_{osc}}\right)$$

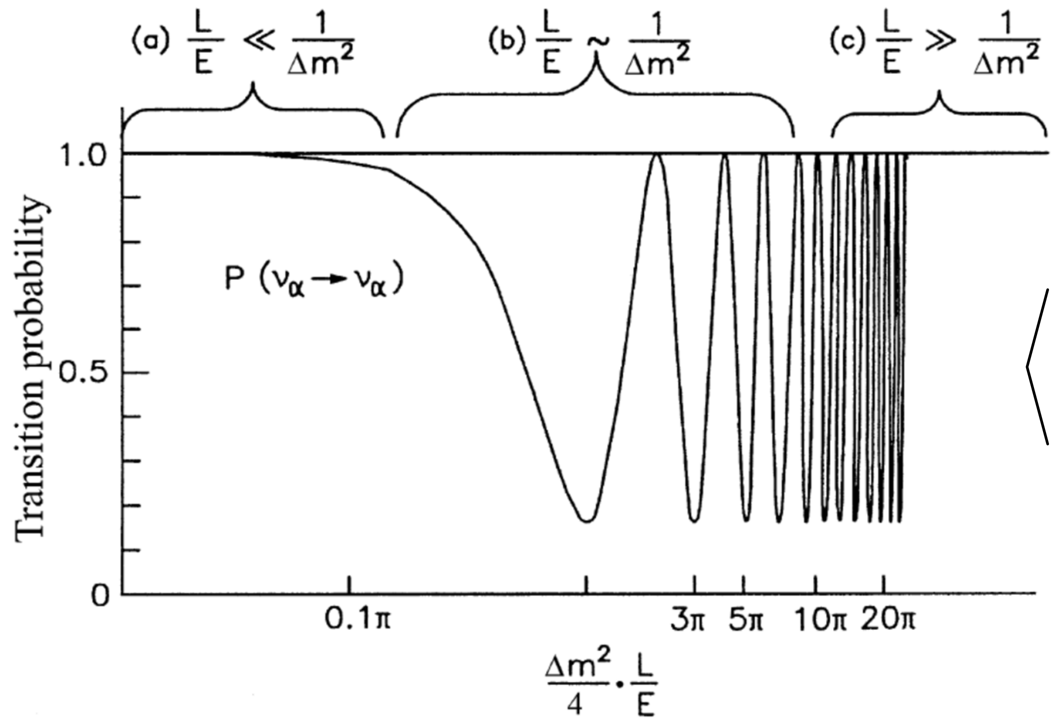


$$\langle \mathcal{P}(\nu_e \rightarrow \nu_e) \rangle = 1 - \frac{1}{2}\sin^2(2\theta)$$

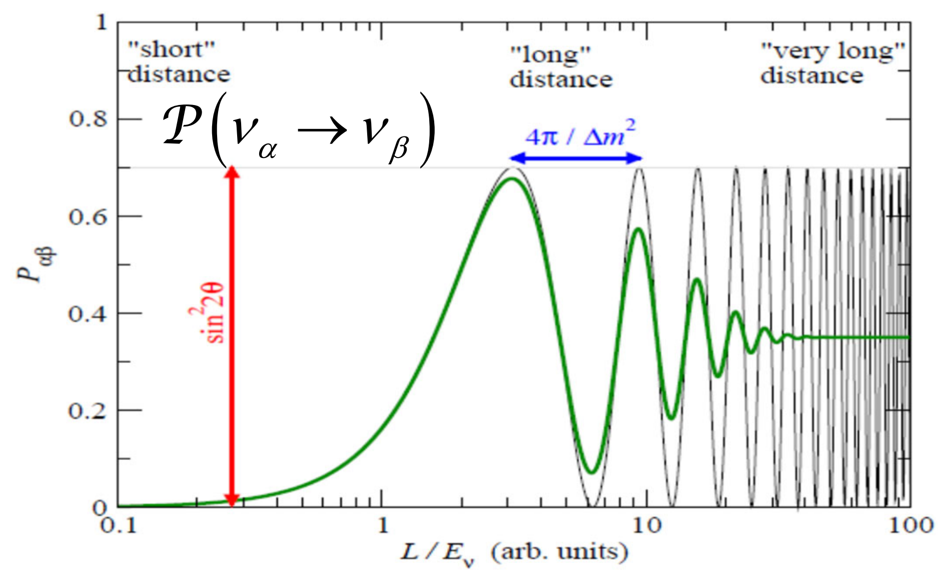
$$\langle \mathcal{P}(\nu_e \rightarrow \nu_\mu) \rangle = \frac{1}{2}\sin^2(2\theta)$$

<u>3 different scenarios:</u>

- $L/E \ll \dfrac{4}{\Delta m^2}$ $\rightarrow$ detector too close to source: oscillation not yet built-up
- $L/E \approx \dfrac{4}{\Delta m^2}$ $\rightarrow$ most sensitive region to observe oscillatory behavior
- $L/E \gg \dfrac{4}{\Delta m^2}$ $\rightarrow$ many oscillation cycles happen: for finite E distribution one measures $\left\langle \sin^2\left(\dfrac{\Delta m^2}{4E}L\right)\right\rangle = \dfrac{1}{2}$



$$\left\langle \sin^2\left(\frac{\Delta m^2}{4E} L\right) \right\rangle = \frac{1}{2}$$



Three neutrino oscillation:

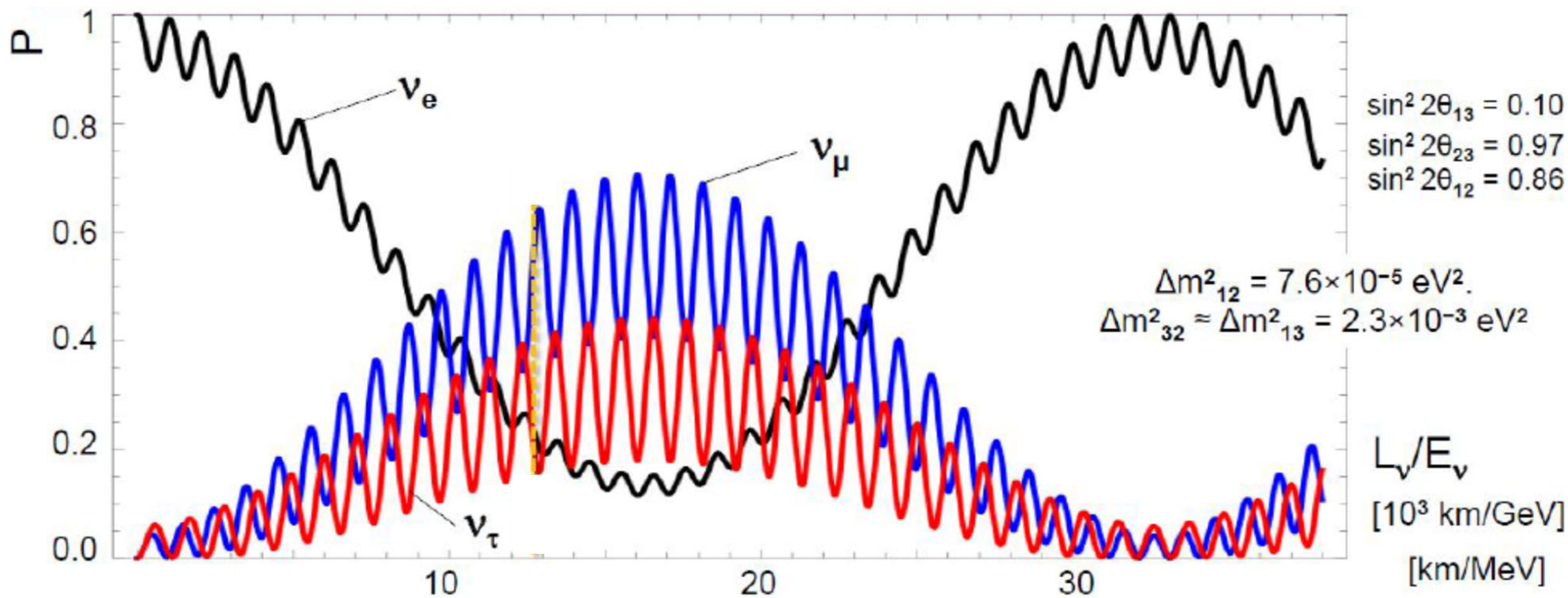
Formula is quite complex

$$\mathcal{P}\left(\nu_\alpha \to \nu_\beta\right) = \left| U_{\alpha 1}U^*_{\beta 1} + U_{\alpha 2}U^*_{\beta 2}\exp\left(-i\frac{\Delta m^2_{21}}{2E}L\right) + U_{\alpha 3}U^*_{\beta 3}\exp\left(-i\frac{\Delta m^2_{31}}{2E}L\right)\right|^2$$

And depend on two Δm^2 and three angles $\theta_{12}, \theta_{23}, \theta_{13}$ and one CP phase. Assume:

$$\Delta m^2_{21} \ll \Delta m^2_{31} \approx \Delta m^2_{32}$$

initial electron neutrino beam.

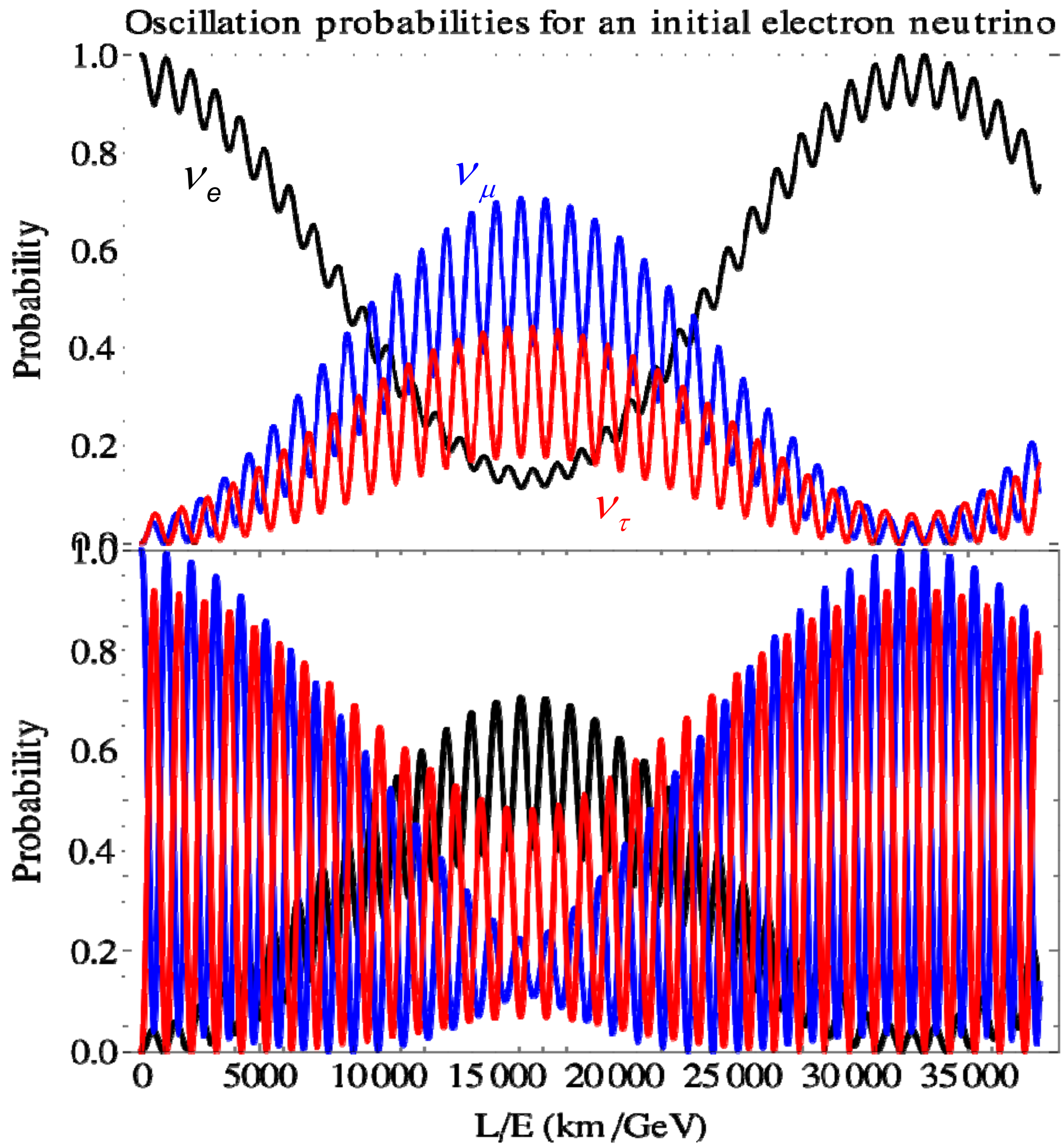


initial electron neutrino

Solar neutrinos

initial muon neutrino

Atmospheric neutrinos



Summary of neutrino oscillations in vacuum:

- if we observe oscillations:
  - $\Rightarrow \Delta m_{ij}^2 \neq 0 \Rightarrow m_i$ oder $m_j \neq 0$
  - U_{PMNS} is non diagonal. ($\rightarrow$ mixing)
- Oscillation provides access to very small Δm_{ij}^2
- Observation of neutrino oscillation in two ways: disappearance of ν_α or appearance of ν_β
- Neutrinos oscillation does not alter the total ν-flux: $\sum_{\nu_\beta} \mathcal{P}\left(\nu_\alpha \rightarrow \nu_\beta\right) = 1$

  However, if some of $\nu\beta$ are "sterile flavors" (no weak IA) it appears that the flux of the total active neutrinos (ν_e , ν_μ ν_τ) is reduced.

5.2 Neutrino oscillation in matter

Neutrino oscillation can be treated by an effective Schrödinger-Eq. (time evolution in the lab frame). For the two-neutrino case one obtains in the flavor basis:

$$i\frac{\partial}{\partial t}\left|\nu(t)\right\rangle = \mathbf{H}\left|\nu(t)\right\rangle \quad \text{with} \quad \left|\nu(t)\right\rangle = \begin{pmatrix} f_e(t) \\ f_\mu(t) \end{pmatrix}$$

$f_{e,\mu}$ is the amplitude for being a ν_e (ν_μ). **H** is a 2×2 matrix (effective Hamiltonian) in ν_e - ν_μ space.

<u>For vacuum</u> one finds for $\mathbf{H}_{vac}$ (up to multiples of the unit matrix **1**):

see textbooks

$$\mathbf{H}_{vac} = \frac{\Delta m^2}{4p}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \underbrace{\left(p + \frac{m_1^2 + m_2^2}{4p}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

results in additional overall phase – can be dropped (not interesting for physics)

with p≈E: $$\mathbf{H}_{vac} = \frac{\Delta m^2}{4E}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$ (in flavor space)

From the mass states ν_1 and ν_2 corresponding to the eigenvalues $\lambda_{1,2} = \mp \frac{\Delta m^2}{4E}$ one gets the **flavor states**:

$$\left|\nu_e\right\rangle = \left|\nu_1\right\rangle \cos\theta + \left|\nu_2\right\rangle \sin\theta$$

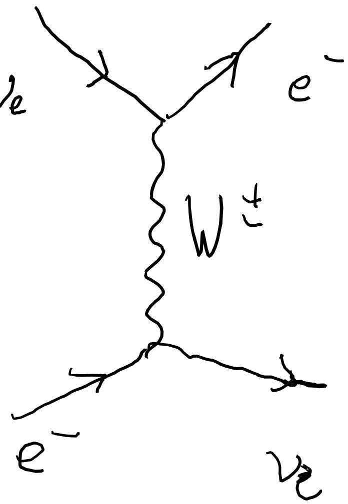
$$\left|\nu_\mu\right\rangle = -\left|\nu_1\right\rangle \sin\theta + \left|\nu_2\right\rangle \cos\theta$$

with $$\left|\nu(t)\right\rangle = \left|\nu_1\right\rangle \exp\left(+i\frac{\Delta m^2}{4E}t\right)\cos\theta + \left|\nu_2\right\rangle \exp\left(-i\frac{\Delta m^2}{4E}t\right)\sin\theta$$

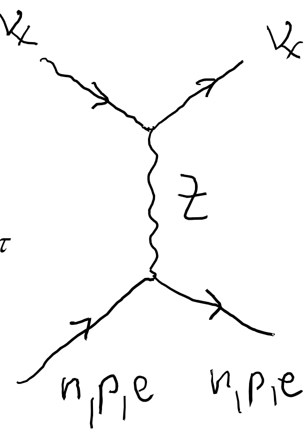
one finds $$\mathcal{P}\left(\nu_e \to \nu_\mu\right) = \left|\left\langle \nu_\mu \middle| \nu(t)\right\rangle\right|^2 = \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2}{4E}L\right)$$ as above

In matter the neutrinos feel an additional potential due to the interaction:

ν_e



ν_e, ν_μ, ν_τ



$$V_W = \pm\sqrt{2}G_F N_e$$

N_e = electron density
(“-”-sign for anti-neutrinos)

$$V_Z = \mp\frac{\sqrt{2}}{2}G_F N_n$$

Effect of electron and proton cancel.
Net effect only from neutron, density N_n

Standard model interactions do not change the neutrino flavor $\rightarrow$ observation of a ν-flavor change when passing matter implies massive neutrinos & mixing!

Hamilitonian $\mathbf{H}_M$ in matter:

$$\mathbf{H}_M = \mathbf{H}_{vac} + V_W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \underbrace{V_Z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

Effect on neutrino oscillation vanishes
$\rightarrow$ additional overall phase

$$\mathbf{H}_M = \mathbf{H}_{vac} + \frac{V_W}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \overbrace{\frac{V_W}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\Longrightarrow \quad \mathbf{H}_M = \frac{\Delta m^2}{4E}\begin{pmatrix} -(\cos 2\theta - x) & \sin 2\theta \\ \sin 2\theta & (\cos 2\theta - x) \end{pmatrix}$$

with $\quad x = \dfrac{V_W/2}{\Delta m^2/4E} = \dfrac{2\sqrt{2}G_F N_e E}{\Delta m^2}$

x is a measure for the importance of the matter effect

One can rewrite the matter Hamiltonian using

$$\Delta m^2_{matter} = \Delta m^2 \cdot \sqrt{\sin^2 2\theta + \left(\cos 2\theta + x\right)^2}$$

$$\sin^2 2\theta_M = \frac{\sin^2 2\theta}{\sin^2 2\theta + \left(\cos 2\theta - x\right)^2}$$

(see testbooks)

$$\Rightarrow \qquad \mathbf{H}_M = \frac{\Delta m_M^2}{4E} \begin{pmatrix} -\cos 2\theta_M & \sin 2\theta_M \\ \sin 2\theta & \cos 2\theta_M \end{pmatrix}$$

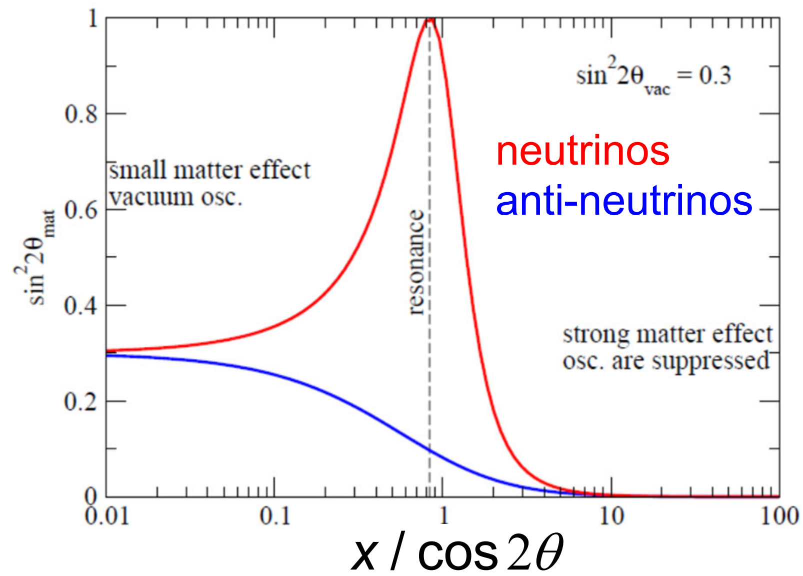
The matter Hamiltonian has the same form as $\mathbf{H}_{\mathbf{vac}}$ $\rightarrow$ eigenstates and mixing described by the expression for the vacuum but replacing Δm^2 and $\sin 2\theta$ by the values in matter: Δm_M^2 and $\sin 2\theta_M$.

$$\left|\nu_e\right\rangle = \left|\nu_1\right\rangle \cos\theta_M + \left|\nu_2\right\rangle \sin\theta_M$$
$$\left|\nu_\mu\right\rangle = -\left|\nu_1\right\rangle \sin\theta_M + \left|\nu_2\right\rangle \cos\theta_M$$

Example 1:

Electron neutrino beam through the mantle of the earth (const. N_e), with a detector at distance of ~1000 km, 2-neutrino approx.:

$$\mathcal{P}_M(\nu_e \to \nu_\mu) = \sin^2 2\theta_M \sin^2\left(\Delta m_M^2 \frac{L}{4E}\right)$$



Matter effect can be very large if

$$\cos 2\theta \approx x = \frac{2V_W E}{\Delta m^2} \quad \Rightarrow \quad \sin^2 2\theta_M = \frac{\sin^2 2\theta}{\sin^2 2\theta + (\cos 2\theta - x)^2} = 1$$

= MSW (Mikheev,Smirnov Wolfenstein) – effect: Dramatic effect since the mixing in matter can be maximal even if it is very small in vacuum.

The matter effect depends on the sign of x= $(2V_W E)/\Delta m^2$, i.e. the sign of $\Delta m_{12}^2 = \Delta m_2^2 - \Delta m_1^2$ and on the sign of the potential V_W which is different for neutrinos and anti-neutrinos.

sign(x)

	ν	$\bar{\nu}$
$m_2 > m_1$	$+$	$-$
$m_1 > m_2$	$-$	$+$

i.e., the sign of the observed matter effect is used to define the mass hierarchy between ν_1 and $\nu_{2,}$ In the 3 neutrino case it can also help to resolve the mass ordering for $\nu_{3.}$

<u>Example 2:</u> Electron neutrino production in core of sun where x $\gg$ 1

$$\sin^2 2\theta_M = \frac{\sin^2 2\theta}{\sin^2 2\theta + (\cos 2\theta - x)^2} \to 0 \quad \Longrightarrow \quad \theta_M \to \frac{\pi}{2}$$

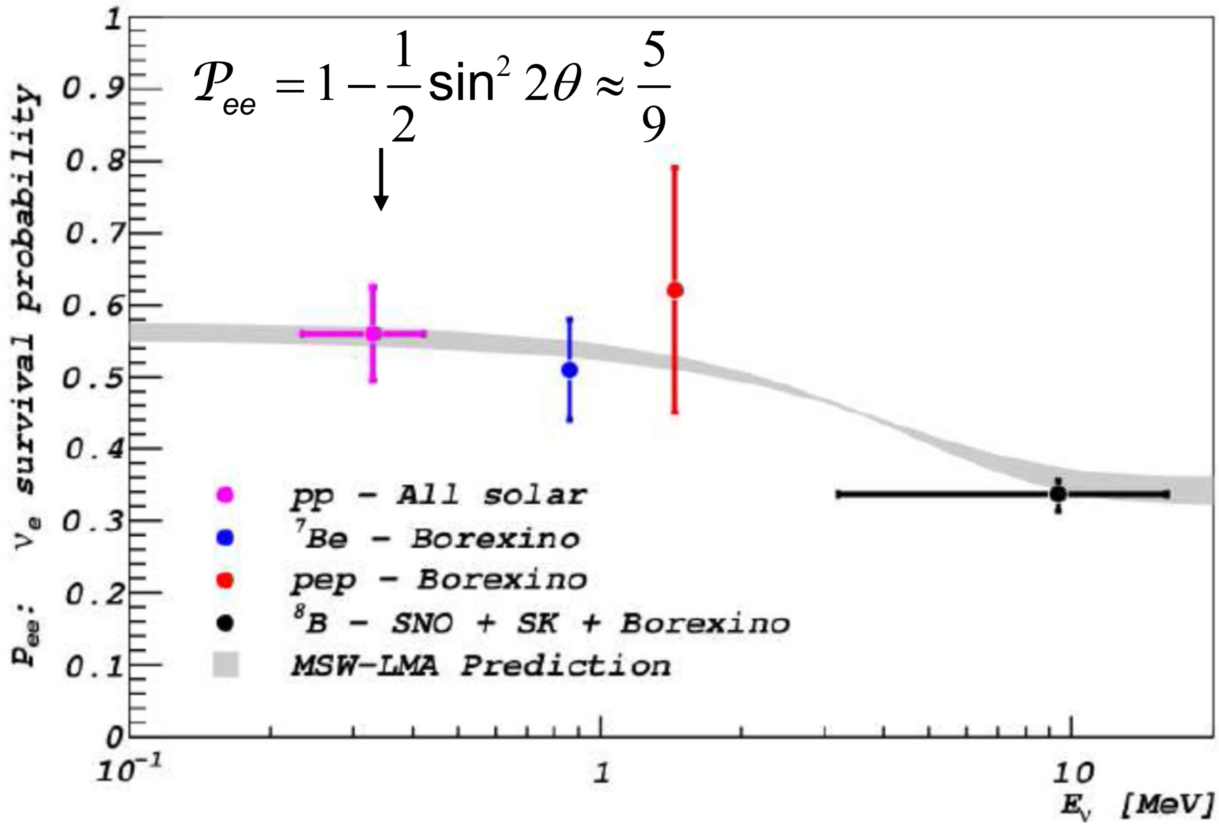
$$\Longrightarrow \quad |\nu_e\rangle = |\nu_1\rangle \cos\theta_M + |\nu_2\rangle \sin\theta_M = |\nu_2\rangle$$

When the density profile changes adiabatically the neutrino stays a in pure mass state $\nu_{2,M}$ also at the surface of the sun when entering the vacuum to travel to the earth: no flavor oscillation. We will measure a pure ν_2 state on earth (which is a composition of ν_e and ν_μ (or $\nu_x \approx \sqrt{\frac{1}{2}}\left(\nu_\mu + \nu_\tau\right)$)

$$|\nu_2\rangle = \cos\theta |\nu_e\rangle + \sin\theta |\nu_\mu\rangle$$

Survival probability for electron neutrinos from the sun:

$$\mathcal{P}_{ee} = 1 - \frac{1}{2}\sin^2 2\theta \approx \frac{5}{9}$$



$$\mathcal{P}_{ee} = \sin^2\theta \approx \frac{1}{3}$$

Survival probability smaller 0.5 possible!

Small matter effect

large matter effect

$$|\nu_2\rangle = \cos\theta|\nu_e\rangle + \sin\theta|\nu_\mu\rangle$$