Repetition:

Yukawa interaction between fermions and Higgs field:

$$\mathcal{L}_{Yukawa} = -\sum_{i,j}\left\{ Y_U^{ij}\bar{Q}_L^i U_R^j \tilde{H} + Y_D^{ij}\bar{Q}_L^i D_R^j H + Y_E^{ij}\bar{L}_L^i E_R^j H + h.c. \right\}$$

$$\mathcal{L}_{Mass} = -\frac{v}{\sqrt{2}}\left\{ \bar{U}_L^i Y_U^{ij} U_R^j + \bar{D}_L^i Y_D^{ij} D_R^j + \bar{E}_L^i Y_E^{ij} E_R^j + h.c. \right\}$$

Yuakwa matrices are in general non-diagonal: terms in Lagrangian cannot be interperted as mass terms – requires diagonalization.

For quarks: $U_R \to V_{u_R} u_R \quad U_L \to V_{u_L} u_L \quad D_R \to V_{d_R} d_R \quad D_L \to V_{d_L} d_L$

$$\mathcal{L}_{Gauge}^{CC} \supset \frac{g}{\sqrt{2}}\left\{ \bar{u}_L \underbrace{V_{u_L}^{\dagger} V_{d_L}}_{V_{CKM}} \gamma_\mu W^{+\mu} d_L + \bar{d}_L \underbrace{V_{d_L}^{\dagger} V_{u_L}}_{V_{CKM}^{\dagger}} \gamma_\mu W^{-\mu} u_L \right\}$$

For leptons: Without loosing an generality one can choose $Y_E$ diagonal as long as neutrisnos are massless.

Quark flavor structure is already imprinted in the Yuakawa coupling to Higgs

# 4. CKM-Matrix and unitary relations

The quark mixing (CKM) matrix has 4 physical parameters, which can be expressed as 3 angles and 1 phase. A common parameterization is given by the subsequent application of 3 rotations (one w/ a phase factor):

$$\mathbf{V}_{\mathrm{CKM}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{13} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-\mathrm{i}\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta_{13}} & c_{23}c_{13} \end{pmatrix}$$

where $s_{12} = \sin\theta_{12}$ , $s_{13} = \sin\theta_{13}$ , $s_{23} = \sin\theta_{23}$ etc. $\delta$ is the phase.
$c_{12} = \cos\theta_{12}$

Remark: One could think of many other parameterization – in particular where to put the phase - however, this is not going to change the physics as only phase differences are observable.

The strength of the couplings $V_{ij}$ exhibit a hierarchy (from experiment):

$$V_{CKM} = \begin{pmatrix} 1 & \varepsilon & \varepsilon^3 \\ \varepsilon & 1 & \varepsilon^2 \\ \varepsilon^3 & \varepsilon^2 & 1 \end{pmatrix} \qquad \text{with} \quad \varepsilon \approx O\left(10^{-1}\right)$$

This pattern motivated L. Wolfenstein to parametrize the CKM matrix in powers of $\lambda = \sin\theta_{12} \approx 0.22 \rightarrow$ very useful for quantitative discussion of quark mixing effects.

$$V = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)\ .$$

With parameters $\lambda = s_{12}$ $\quad A\lambda^2 = s_{23}$ $\quad A\lambda^3\left(\rho + i\eta\right) = s_{12}e^{-i\delta}$ $\quad$ A, ρ, η ≈ O(1).

Up to O( $\lambda^3$) only the matrix elements $V_{td}$ and $V_{ub}$ are complex.

Matrix element $V_{ts}$ becomes complex only at O( $\lambda^4$).

The 2×2 (d,s) sub-matrix is only complex in higher order (reflects that the Cabibbo matrix was real).

## Unitarity relation:

From the unitarity conditions $V^{\dagger}V = 1$ and $VV^{\dagger} = 1$ one obtains the following 12 equations:

$$V^*_{\substack{ud\\cd\\td}} V_{\substack{ud\\cd\\td}} + V^*_{\substack{us\\cs\\ts}} V_{\substack{us\\cs\\ts}} + V^*_{\substack{ub\\cb\\tb}} V_{\substack{ub\\cb\\tb}} = 1$$

- Universality of weak interaction:
  u couples to the sum of d-type quark in (same way as c and t)

$$V^*_{\substack{ud\\us\\ub}} V_{\substack{ud\\us\\ub}} + V^*_{\substack{cd\\cs\\cb}} V_{\substack{cd\\cs\\cb}} + V^*_{\substack{td\\ts\\tb}} V_{\substack{td\\ts\\tb}} = 1$$

- The sum of u→d, u→s, u→b and sum of u→d, c→d, t→d add-up to 1
  → no space for other decays.

$$V^*_{\substack{ud\\ub\\cd}} V_{\substack{cd\\td\\td}} + V^*_{\substack{us\\us\\cs}} V_{\substack{cs\\ts\\ts}} + V^*_{\substack{ub\\ub\\cb}} V_{\substack{cb\\tb\\tb}} = 0 \quad \leftarrow \text{B}$$

$$V_{\substack{ud\\ud\\us}} V^*_{\substack{us\\ub\\ub}} + V_{\substack{cd\\cd\\cs}} V^*_{\substack{cs\\cb\\cb}} + V_{\substack{td\\td\\ts}} V^*_{\substack{ts\\tb\\tb}} = 0 \quad \leftarrow \text{A}$$

- These relations describe triangle relations in the complex plane.
- It is very instructive to study two of these relations in more detail.

A: $V_{ud}V_{ub}^{*} + V_{cd}V_{cb}^{*} + V_{td}V_{tb}^{*} = 0$

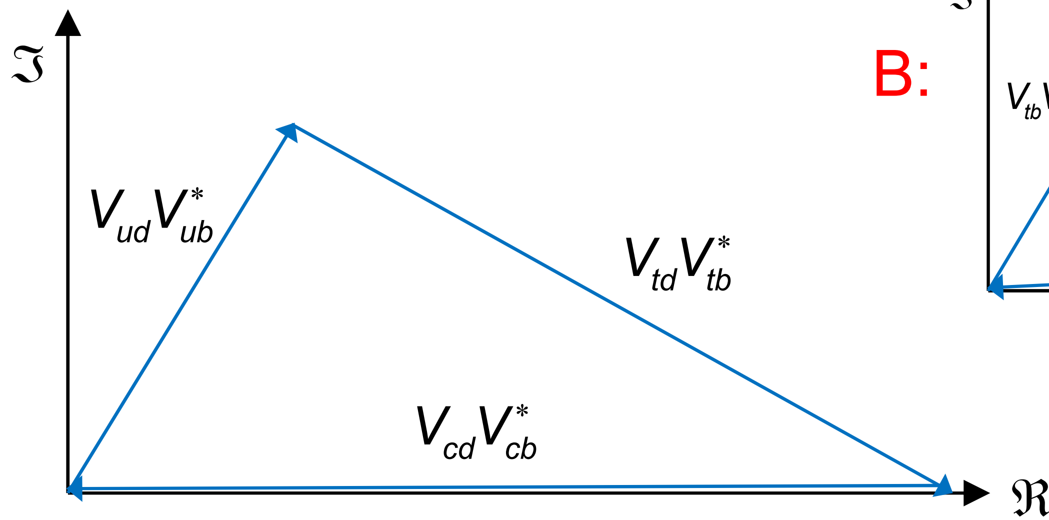
$O\left(\lambda^3\right) \quad O\left(\lambda^3\right) \quad O\left(\lambda^3\right)$

B: $V_{ud}^{*}V_{td} + V_{us}^{*}V_{ts} + V_{ub}^{*}V_{tb} = 0$

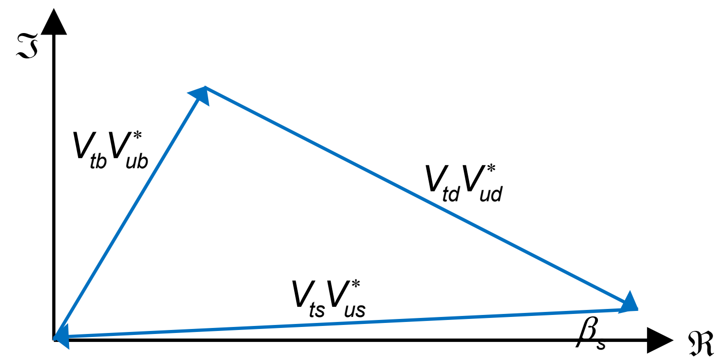
$O\left(\lambda^3\right) \quad O\left(\lambda^3\right) \quad O\left(\lambda^3\right)$

These are the only triangle relations where all sides have approx. the same size $O\left(\lambda^3\right)$ using the Wolfenstein parameter λ. One can draw the relations as triangles in the complex plane.

A:




B:

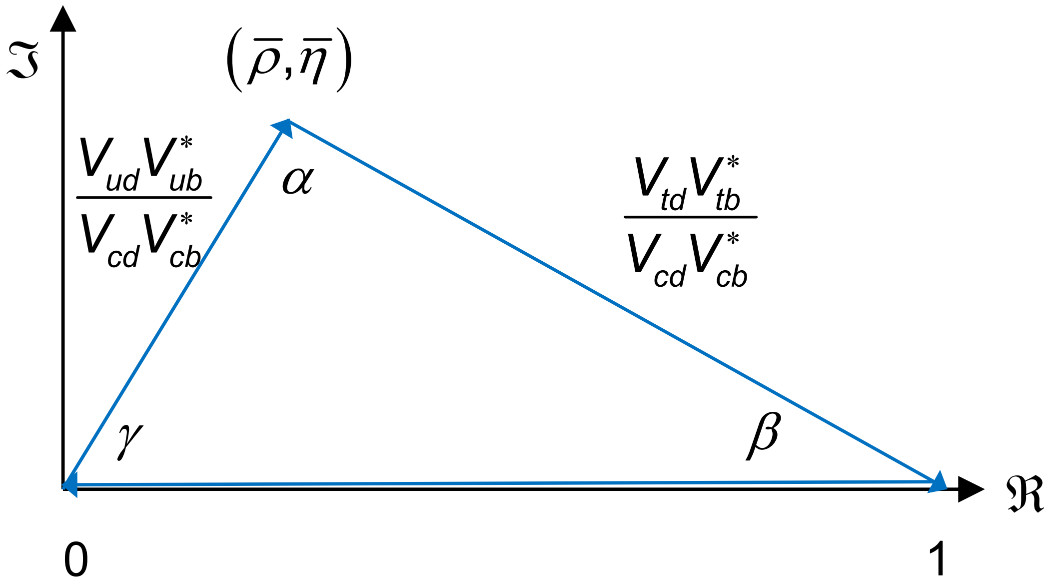



(slightly tilted as $V_{ts}$ has a small imaginary part at $O(\lambda^4)$)

The other 4 relations describe triangles which are squashed.

By dividing in A all sides by $V_{cd}V_{cb}^*$ one obtains the Unitarity Triangle UT

A:




The apex of the triangle $(\bar{\rho},\bar{\eta})$ is given by: $\bar{\rho} + i\bar{\eta} = \frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}$

In terms of the Wolfenstein parameters the apex is given by:

$$\bar{\rho} = \rho\left(1-\tfrac{1}{2}\lambda^2\right) + O\left(\lambda^4\right) \qquad \bar{\eta} = \eta\left(1-\tfrac{1}{2}\lambda^2\right) + O\left(\lambda^4\right)$$

For the three angles one finds:

$$\alpha = \arg\left[-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right] \quad \beta = \arg\left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right] \quad \gamma = \arg\left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right]$$

With the used phase convention and at $O\left(\lambda^3\right)$ only $V_{td}$ and $V_{ub}$ have an imaginary component. In $O\left(\lambda^4\right)$ also $V_{ts}$ is imaginary.

$$\beta = \arg\left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right] \qquad \gamma = \arg\left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right]$$

One finds: $\beta \approx -\arg\left(V_{td}\right)$ and $\gamma \approx -\arg\left(V_{ub}\right)$ and $\beta_s \approx -\arg\left(V_{ts}\right) \approx 0$
(from the triangle B)

One can rewrite the CMK matrix (approximation to $O(\lambda^3)$).

$$V_{CKM} = \begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}|\cdot e^{-i\gamma} \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}|\cdot e^{-i\beta} & |V_{ts}| & |V_{tb}| \end{pmatrix}$$

$$|V_{ts}|\cdot e^{-i\beta_s}$$

Attention:
The specific appearance of the phases in the CKM matrix is a convention. However, the angles $\alpha$, $\beta$, $\gamma$, as defined above (p. 23) are "rephasing invariant" parameters, independent of the specific CKM parameterization.

Another “rephasing” invariant variable is the Jarlskog invariant J:

$$\Im\left(V_{ij}V_{kl}V_{il}^{*}V_{kj}^{*}\right) = J \cdot \sum_{m,n} \varepsilon_{ikm}\varepsilon_{j\ln}$$

e.g.: $$J = \Im\left(V_{ud}V_{cb}V_{ub}^{*}V_{cd}^{*}\right)$$

The area of the unitarity triangles is given by: $\frac{1}{2}J$

In Wolfenstein param.: $J \approx A^2\lambda^6\eta \approx 3\cdot 10^{-5}$

PDG parameters: $J = c_{12}c_{13}^2c_{23}s_{12}s_{13}s_{23}\sin\delta$

The Jarlskog invariant is a measure of the CP violation in the Standard Model. It appears in all CP violating effects. It is zero if one of the mixing angles $\theta_{ij}$ vanishes or if the phase $\delta$ is zero or $\pi$.

CP violation in the Standard Model:

$$\mathcal{L} \sim \underline{\bar{u}_L V_{ub} \gamma_\mu b_L W^{+\mu}} + \underline{\bar{b}_L V_{ub}^* \gamma_\mu u_L W^{-\mu}}$$

↓ CP equivalent to T transformation.

$$\mathcal{L}_{CP} \sim \underline{\bar{b}_L V_{ub} \gamma_\mu u_L W^{-\mu}} + \underline{\bar{u}_L V_{ub}^* \gamma_\mu b_L W^{+\mu}}$$

CP is conserved if $V_{ub} = V_{ub}^*$ (here) or, more general, if CKM-phase $\delta = 0$

In the Standard Model the CKM phase $\delta$ is the only source of CP violation.

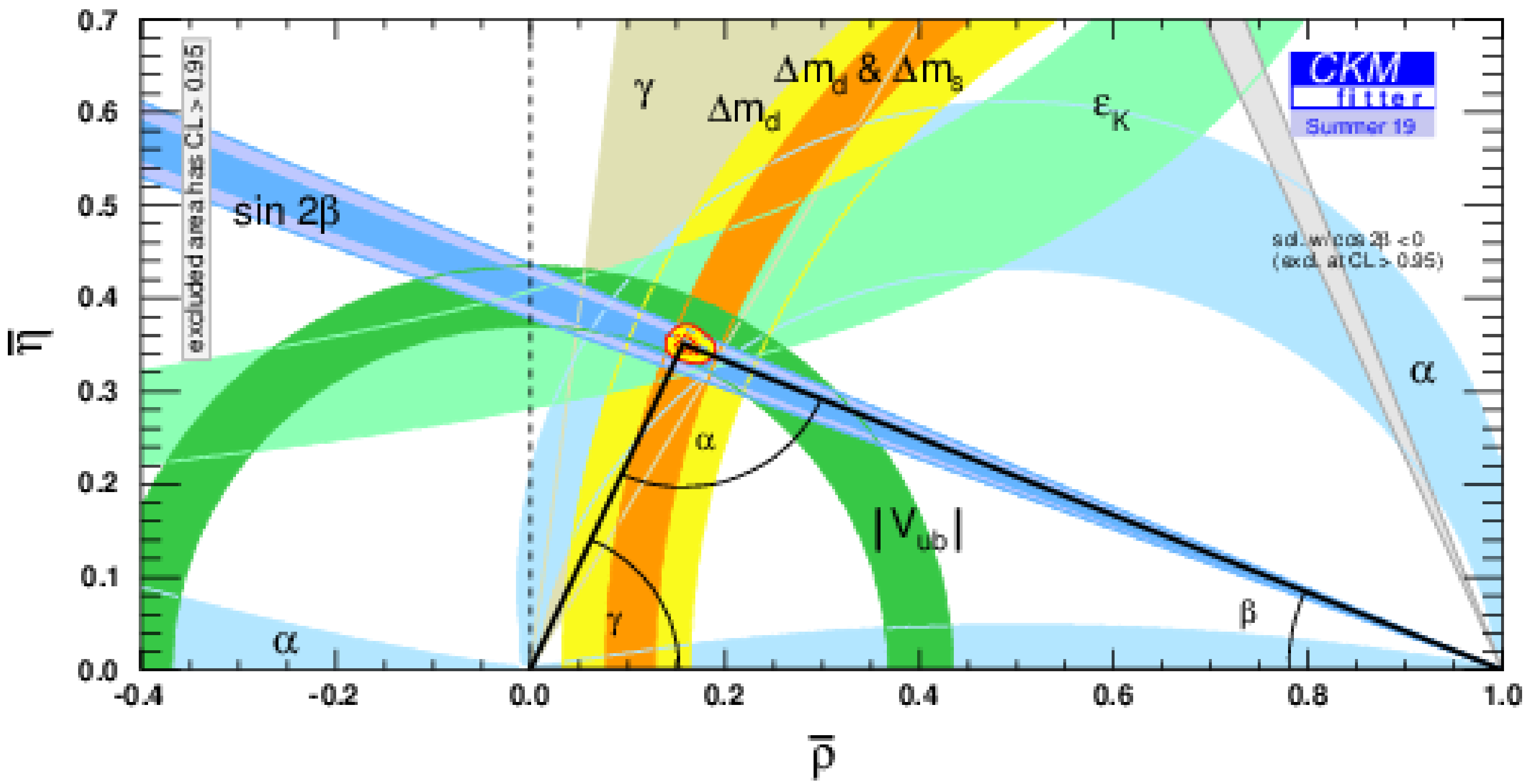
CP violation in weak interaction thus not only requires at least three generations of quarks but also mixing among the different flavors. Moreover CP violation requires the quarks with different flavors but equal gauge quantum numbers to have non-degenerate masses: Why?

$$\Delta m_{ij}^2 \equiv m_i^2 - m_j^2 \neq 0$$

(otherwise one could absorb the CP phase by a redefinition of the quark fields)

Experimental status of the Unitarity Triangle:

*http://ckmfitter.in2p3.fr/*




With the exception of $\varepsilon_{K,}$ which is measured from CP violation in the neutral kaon system, all experimental constraints come from the precise measurements of B mesons: CP violation, mixing and semi-leptonic decays.

## What did we learn?

Unitarity relation of the CKM-matrix can be depicted graphically (triangles) in the complex plane: There are two UT which have approximatly equal sides – the other 7 triangles are squashed.

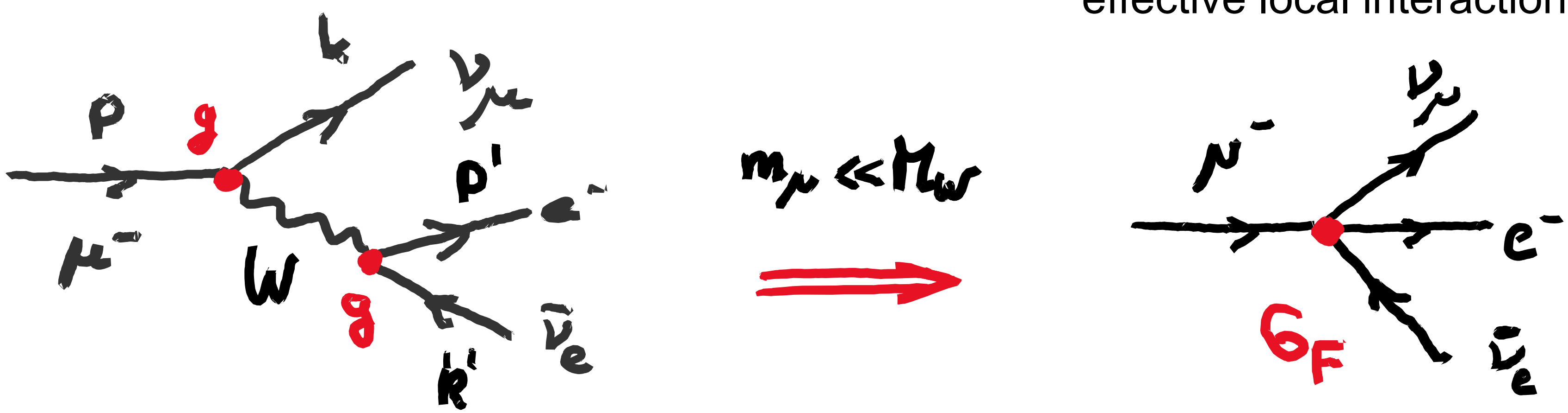
The triangle are all have the same area J/2 – where J is the Jarlskog invariant. J is a measure of the size of CP violation in the Standard Model.

The experimental validation establishes the Unitarity of the CKM matrix. In the validation B mesons play an impotant role.

# 4. Weak charged-current decays

Muon decay: (simplest charged current decay)

effective local interaction



$$\mathcal{M} \sim g \cdot J^{\alpha}_{muon} \cdot \left( \frac{g_{\alpha\beta}}{M_W^2 - q^2} \right) \cdot g \cdot J^{\beta}_{electron} \quad \Rightarrow \quad \mathcal{M} \sim \frac{G_F}{\sqrt{2}} \cdot J^{\alpha}_{muon} \cdot J_{electron,\beta}$$

g is the $SU(2)_L$ coupling

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}$$

Writing-out the fermion currents one obtains:

$$\mathcal{M} \sim \frac{4G_F}{\sqrt{2}} \cdot \left( \overline{\nu}_{\mu,L} \gamma^{\alpha} \mu_L \right) \left( \overline{e}_L \gamma_{\alpha} \nu_{e,L} \right)$$

Squaring the matrix element and summing over all the spin configurations in the final state and averaging over the spins of the initial state one obtains (see e.g. text books by Halzen&Martin or Thomson):

$$\overline{|\mathcal{M}|^2} \sim 64 G_F^2 \cdot (k \cdot p')(k' \cdot p)$$

Taking the phase-space factor into account one finally obtains the decay width $\Gamma$ ( $=1/\tau$ ) of the muon:

$$\Gamma = \frac{1}{\tau} = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

With the mass of the muon (106 MeV) and the muon lifetime (2.2 $\mu$s) one obtains:

$$G_F = 1.16 \cdot 10^{-5} \text{GeV}^2$$

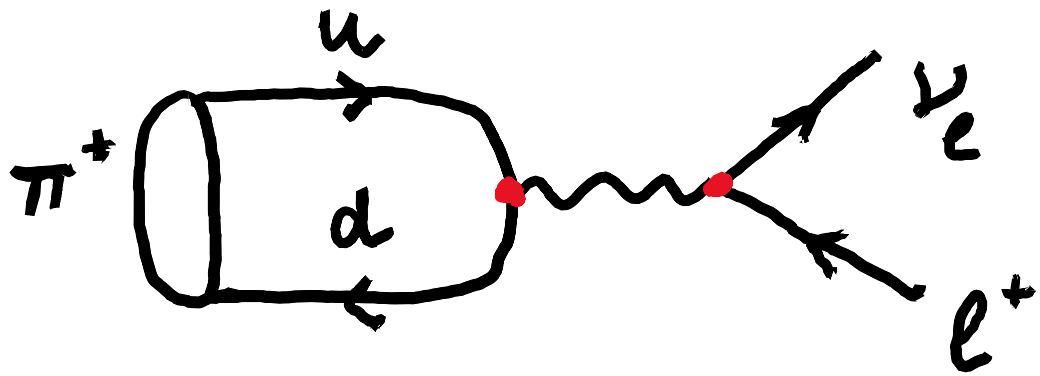
$G_F$ was originally introduced by Fermi to describe the “energy independent” nuclear beta decay within an effective 4-fermion theory. In the Standard Model

$$G_F = \sqrt{2} \frac{g^2}{8 M_W^2}$$

with g being the $SU(2)_L$ coupling

Pion decay:

$$\pi^+ \to \mu^+ \nu_\mu$$
$$\pi^+ \to e^+ \nu_e$$




Textbook example of the helicity suppression in weak decays:

From phase space considerations one would expect

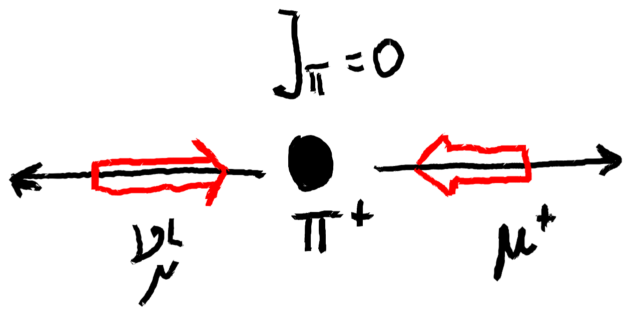
$$\mathcal{B}\left(\pi^+ \to e^+ \nu_e\right) \gg \mathcal{B}\left(\pi^+ \to \mu^+ \nu_\mu\right)$$

However one measures:

$$\frac{\mathcal{B}\left(\pi^+ \to e^+ \nu_e\right)}{\mathcal{B}\left(\pi^+ \to \mu^+ \nu_\mu\right)} \approx \left(1.230 \pm 0.004\right) \cdot 10^{-4}$$

(for practical reasons charged pions decay only into μ.)

Suppression can be easily understood from angular momentum conservation:




Reminder: Helicity

$$H = \frac{\vec{s}\cdot\vec{p}}{|\vec{p}|} \qquad H = \pm\tfrac{1}{2}$$

Due to angular momentum conservation and the fact that there are only LH neutrinos the $\mu^+$ must have negative helicity. In the zero-mass limit this is not possible: the weak interaction only couples LH fermions and RH anti-fermion.

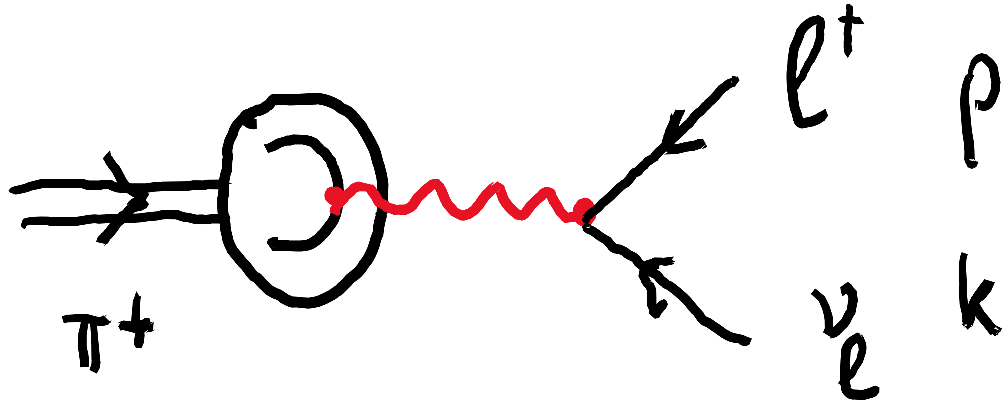
However for massive leptons the production of "wrong helicity states" are possible - helicity suppression:

$$\frac{\mathcal{B}\left(\pi^+ \to e^+\nu_e\right)}{\mathcal{B}\left(\pi^+ \to \mu^+\nu_\mu\right)} = \underbrace{\frac{m_e^2}{m_\mu^2}}_{\text{Helicity suppression}} \cdot \underbrace{\frac{\left(1-\frac{m_e^2}{m_\pi^2}\right)^2}{\left(1-\frac{m_\mu^2}{m_\pi^2}\right)^2}}_{\text{Phase space}} \approx 1.275\cdot 10^{-4}$$

(if one considers in addition radiative corrections, the experimental value is reproduced)

## Weak decay of the bound quark-state:

Although quarks are the fundamental particles, it is the pion which is participating in the weak interaction and which enters as the asymptotic particle:




$$M \sim \frac{G_F}{\sqrt{2}} \cdot (\pi)_\mu \, (\text{lepton})^\mu$$

With the pion spin $J_\pi=0$ the only relevant 4-vector is the momentum transfer $q^\mu = p^\mu + k^\mu$ to the lepton system:

$$\mathcal{M} = -i\sqrt{2} G_F V_{ud} \cdot f_\pi q_\mu \cdot \overline{\ell}_L \gamma^\mu \nu_\ell$$

where **$f_\pi$** is the pion decay constant.

Decay constant:

$f_\pi$ is the value of a pion-to-vacuum transition described by the hadronic matrix element:

$$\left\langle 0 \left| \bar{u}_L \gamma^\alpha d_L \right| \pi\left(q_\pi\right) \right\rangle$$

In the "chiral-limit" (limit of massless quarks):

$$f = f_\pi = f_K$$

Experimentally one finds:

$$\pi \to \mu\nu: \quad f_\pi \approx 131\,\text{MeV}$$

$$K \to \mu\nu: \quad f_K \approx 164\,\text{MeV}$$

The difference between $f_\pi$ and $f_K$ demonstrates that the $SU(3)_V$ valence quark symmetry is broken by the strange quark mass.

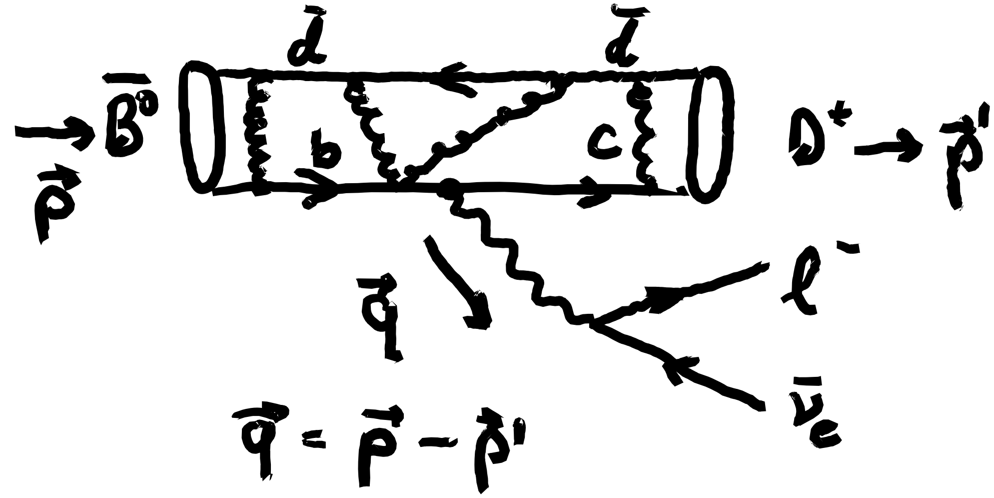
<u>Pion-decay:</u>

$$\Gamma\left(\pi^+ \to \ell^+ \nu_\ell\right) = \frac{G_F^2}{8\pi} \left|V_{ud}\right|^2 f_\pi^2 \cdot m_\ell^2 m_\pi \left(1 - \tfrac{m_\ell^2}{m_\pi^2}\right)^2$$

<u>Semi-leptonic meson decays:</u>

Example:

$$\overline{B}^0 \rightarrow D^+ \ell^- \nu_e$$




Matrix element can be expressed in terms of Lorentz-invariant form-factors

$f_+\left(q^2\right)$ and $f_-\left(q^2\right)$ with:

$$\left\langle D\left(p'\right)\middle|\overline{c}_L\gamma_\alpha b_L\middle|\overline{B}^0\left(p\right)\right\rangle = f_+\left(q^2\right)\left(p+p'\right)_\alpha + f_-\left(q^2\right)\left(p-p'\right)_\alpha$$

The invariant transition amplitude reads:

$$\mathcal{M} \sim \sqrt{2}G_F V_{cb} f_+\left(q^2\right)\left(p+p'\right)^\alpha \overline{\ell}_L\left(p_\ell\right)\gamma_\alpha \nu_\ell\left(p_\nu\right)$$

($f_-(q^2)$ does not contribute in the limit $m_l = 0$)

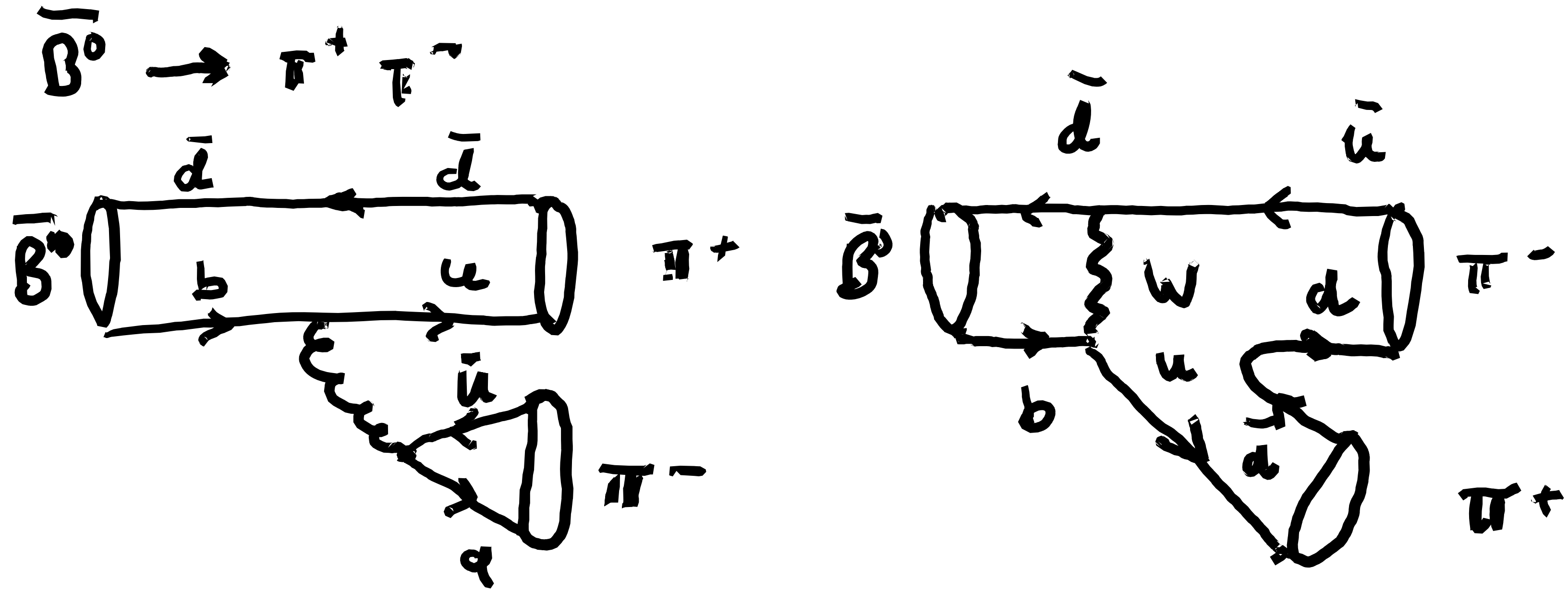
Differential decay rate

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2}{192\pi^3}\frac{\left|V_{cb}\right|^2}{m_B^3}\left|f_+\left(q^2\right)\right|^2\left[\left(q^2 - m_B^2 - m_D^2\right)^2 - 4m_B^2 m_D^2\right]^{\frac{3}{2}}$$

$$0 \le q^2 \le (m_B - m_D)^2$$

(form-factor $f_+$ can be calculated using lattice gauge theory)

Hadronic meson decays:




Difficult as there are strong interactions between final state and initial state.

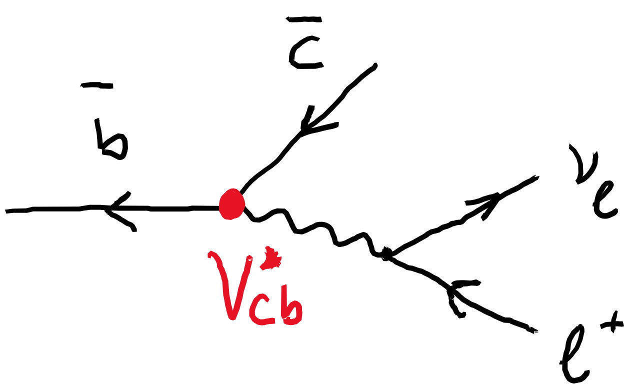
For $M_W \gg \sqrt{q^2} > \Lambda_{QCD}$ a separation of QCD and electroweak physics is possible $\rightarrow$ treatment within an effective theory ansatz: similar to Fermi's treatment of the beta decays, the weak interaction between the quarks is expressed by an effective operator coupling the in and out-going fermions.

## Higher-order flavor-changing neutral-current decays (FCNC):

Neutral currents due to Z-boson or photon interactions are flavor conserving (flavor diagonal) in the Standard Model.

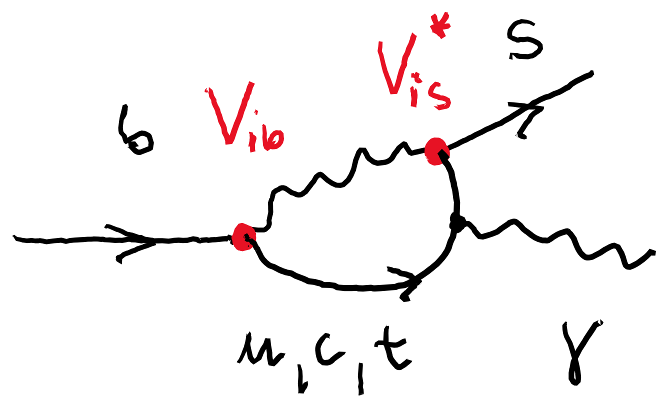
Flavor-changing neutral current decays however are still possible considering higher-order quantum corrections. Consider the following two B-decays:

$$\mathcal{B}\left(B^+ \to \bar{D}^0 \ell^+ \nu_\ell\right) \approx 2 \cdot 10^{-2}$$




$$\mathcal{M} \sim g^2 V_{cb}^* \sim g^2 \lambda^2$$

$$\mathcal{B}\left(B \to X_s \gamma\right)_{E_\gamma > 1.6\,\mathrm{GeV}} \approx 4 \cdot 10^{-4}$$




$$\mathcal{M} \sim e \frac{g^2}{16\pi^2} \sum_{i=u,c,t} V_{ib} V_{is}^* F\left(\frac{m_i^2}{M_W^2}\right)$$

Loop-function F has a mass independent part F(0) and a mass dependent part:

$$F\left(\frac{m_i^2}{M_W^2}\right) = F(0) + \tilde{F}\left(\frac{m_i^2}{M_W^2}\right)$$

Mass independent part vanishes due to CKM unitarity:

$$\mathcal{M} \sim F(0) \underbrace{\sum_{i=u,c,t} V_{ib} V_{is}^*}_{=0} + \sum_{i=u,c,t} V_{ib} V_{is}^* \tilde{F}\left(\frac{m_i^2}{M_W^2}\right)$$

With $V_{tb} V_{ts}^* = -\sum_{i=u,c} V_{ib} V_{is}^*$ one obtains:

$$\mathcal{M} \sim \sum_{i=u,c} V_{ib} V_{is}^* \left(\tilde{F}\left(\frac{m_i^2}{M_W^2}\right) - \tilde{F}\left(\frac{m_t^2}{M_W^2}\right)\right) \approx -\left(\underset{\lambda^4}{V_{ub} V_{us}^*} + \underset{\lambda^2}{V_{cb} V_{cs}^*}\right) \tilde{F}\left(\frac{m_t^2}{M_W^2}\right)$$

$$\tilde{F}\left(\frac{m_i^2}{M_W^2}\right) \approx \frac{m_i^2}{M_W^2} + O\left(\frac{m_i^2}{M_W^2}\right)$$

Note that for $m_u = m_c = m_t$ , $\mathcal{M} = 0$ . This is true for any FCNC process. This is usually called "GIM mechanism" (Glashow, Iliopoolous, Maiani).

Together with the CKM suppression ($\sim\lambda^2$) it explains the strong suppression of FCNC effects in the Standard Model. For B-decays it is the top-quark loop with large mass which deactivates the GIM mechanism and contributes to the effects.

## What did we learn?

To describe weak meson decays need to introduce form-factors to account for the hadronic matrix elements:

$$\left\langle 0 \middle| \bar{u}_L \gamma^\alpha d_L \middle| H(q) \right\rangle \qquad \left\langle h(q) \middle| \bar{u}_L \gamma^\alpha d_L \middle| H(q) \right\rangle$$

### FCNC decays:

GIM suppression: unitarity of CKM matrix would lead to vanishing loop contributions in case the quarks have the same masses or vanishing masses.

Deactivation of GIM suppression in B-decays: large top mass leads to a strong contribution even if the loop is CKM suppressed.